NUMBER FRACTION FOR THE PRODUCTS OF POWERS OF PRIMES

We have shown in earlier notes that any positive integer has associated with it the number fraction-

\[ f(N) = \frac{\sigma(N) - (N + 1)}{N}, \]

where \( \sigma(N) \) is the familiar sigma or divisor function representing the sum of all divisors of \( N \) including 1 and \( N \). The number fraction has an advantage over the divisor function in that \( f(N) = 0 \) for all cases where \( N \) is a prime \( p \) and the \( N \) term in its denominator insures that it is a very slow increasing function of \( n \) as \( N \) gets large. We want here to work out the value of the number fraction \( f(p_1^{n_1} p_2^{n_2} p_3^{n_3} \ldots) \) where \( p_1, p_2, p_3 \) etc are primes taken to the integer powers \( n_1, n_2, n_3, \ldots \) The reason for doing this is that one knows any positive integer \( N \) can be represented by the products of primes taken to specified exponents. So once we have found \( f(p_1^{n_1} p_2^{m_2} p_3^{n_3} \ldots) \) the value of \( f(N) \) will be known. Let us begin with a semi-prime \( N = pq \) where \( p \) and \( q \) are primes taken to the first power. Writing out the number faction in long hand we find-

\[ pq \]

so that \( f(77) = f(7 \cdot 11) = (7 + 11)/77 = 18/77 \). Next look at the case of a three prime product \( N = pqr \). The simplest way here to get a general formula is to take the elementary case \( p = 2, q = 3, \) and \( r = 5 \). This means \( N = 30 \) and its \( f(N) \) reads \((2 + 3 + 5 + 6 + 10 + 15)/30 \). Now going backwards we see that \( 2 = p, 3 = q, 5 = r, 6 = pq, 10 = pr, \) and \( 15 = qr \). So we conclude that the following generic formula holds-

\[ f(pqr) = \frac{p + q + r + (pq + pr + qr)}{pqr}, \]

where \( p, q, \) and \( r \) are different primes. Thus the number 125 = 3 · 5 · 7 has \( f(125) = f(3 \cdot 5 \cdot 7) = (3 + 5 + 7)/(15 + 21 + 35)/125 = 86/125 \). Introducing the product of four primes taken to the first power each, we find-

\[ f(pqrs) = \frac{(p + q + r + s) + (pq + pr + qr + ps + qs + rs) + (pqr + pqs + prs + qrs)}{pqrs} \]

Again none of the primes can repeat themselves.

It is clear from these results that the formulas for the Number Fraction can get rather complicated as the number of primes in the product increases. The pattern is, however, recognizable with terms in the numerator being the prime components taken in groups of \( 1, 2, 3, \ldots (n-1) \) at a time and added together. Thus-
The number of terms in each group within the numerator is just equal to the binomial coefficient \( C_{n,m} = \frac{n!}{m!(n-m)!} \), with \( n \) being the total number of primes representing \( N \) and \( m \) the number of primes which are being taken in a group. Thus four terms \( 3+5+7+11=26 \) make up the first group of \( 4!/[1!(3!)]=4 \). The number of terms in the second group is \( 4!/[2!(4-2)!]=6 \). An important observation involving the above formulas for \( f(pq) \), \( f(pqr) \) and \( f(pqrs) \) is that they will not work if any of the primes are identical. For example if we take \( f(2\cdot2\cdot5)=f(20)=17/20 \) while the formula for \( f(pqr) \) yields the incorrect answer \( 33/20 \).

The next question which arises is what happens to the number fraction when some of the primes entering the definition for \( N \) have higher powers. The answer is not directly obtainable from the above first power formulas but rather requires a re-expansion. We find the following formula for \( f(p^2q) \):

\[
f(p^2q) = \frac{(p + q)(1 + p)}{p^2q}
\]

with \( p \neq q \). By replacing \( p \) by \( q \) and \( q \) by \( p \) in this expression one obtains the formula for \( f(pq^2) \).

Taking \( p=61 \) and \( q=47 \) we get \( f(p^2q)=f(174887)=6696/174887 \). Going one step further we have-

\[
f(p^2q^2) = \frac{(p + q) + (p^2 + q^2) + pq(1 + p + q)}{(pq)^2}
\]

We can also generalize the above to get-

\[
Nf(p^nq^m) = \sum_{k=1}^{n} p^k + \sum_{k=1}^{m} q^k + \sum_{k=1}^{m} (pq)^k F_k(p,q)
\]

We have set \( N=p^nq^m \), assumed that \( n\geq m \) and that \( F_k(p,q) \) represents series in \( p \) and \( q \). This result can be further simplified by using the identity-

\[
\sum_{k=1}^{n} r^k = \frac{r(r^n - 1)}{(r-1)}
\]

Having obtained explicit expressions for the Number Fraction of the product of various powers of primes, we are now in a position to obtain the value \( f(N) \) for any desired positive integer \( N \). Sometimes this will involve additional formulas not given explicitly.
by the above but which are easily derivable. Consider first the number
N=p²q=70699=61² x 19. Here one of has from above that-

\[ f(N) = \frac{(61 + 19)(1 + 61)}{70699} = \frac{4960}{70699} \]

This quotient could of course also have been gotten directly by using our PC and looking
at the divisors of N. This produces \( f(N) = (19 + 61 + 3721 + 1159)/N \) which yields the same
result. The advantage of the type of formulas given above is that they only require a
knowledge of \( p, q, r, \) etc. easily obtained in most computer programs by asking for
\texttt{ifactor(N)}.

As the next number consider \( N=4323 \). Using the \texttt{ifactor(N)} in our MAPLE program
shows-

\[ N=4323=3\cdot11\cdot131 \]

So here we have the primes \( p=3, q=11, \) and \( r=131 \). Substituting into the \( f(pqr) \) formula
we find-

\[ f(4323) = \frac{(3 + 11 + 131) + (33 + 393 + 1441)}{33 \cdot 131} = \frac{2012}{4323} \]

As a third example consider the number \( N=6453 \). An \texttt{ifactor(N)} operation shows this
number to be equivalent to \( 3^3\cdot239 \). We can derive the formula for \( f(N) \) in this case using
the above generalized expression for \( f(p^aq^m) \) on setting \( n=3 \) and \( m=1 \). The resultant
expression reads-

\[ f(p^aq) = \frac{(p + q)(1 + p + p^2)}{p^aq} \]

Substituting in \( p=3 \) and \( q=239 \), we get \( f(6453)=3146/6453 \). You will note that \( f(p^3q) \) and
\( f(p^2q) \) differ only in having an additional \( p \) term in finite geometric series in the
numerator of \( f \). This leads to an obvious generalization for \( f(p^nq) \) after using the sum of
the finite term geometric series. The result reads-

\[ f(p^nq) = \frac{(p + q)(p^n - 1)}{p^nq(p - 1)} \]

Thus \( N=416=2^5\cdot13 \) has \( f(416)=15(31)/416=465/416 \).

Finally we look at \( N=576480=7^8 \). Here the number is equivalent to the eighth power of
just one prime number 7. We have a formula for this type of number developed in an
earlier note. It says essentially that for a prime \( p \) one has-
\[ f(p^n) = \frac{1 - p^{1-n}}{p-1} \]

So for \( N = 7^8 \) we find –

\[ f(7^8) = \frac{7(7^7 - 1)}{6 \cdot 7^8} = \frac{137257}{823543} \]

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