EVALUATION OF NUMBER FRACTIONS

Several years ago we came up with a new point function defined as-

\[ f(N) = \frac{[\sigma(N) - N - 1]}{N} \]

Here \( \sigma(N) \) is the sigma function of number theory representing the sum of all divisors of \( N \). We have termed \( f(N) \) the **Number Fraction**. It has the interesting property that it vanishes whenever \( N \) is a prime, has values greater than zero when \( N \) is a composite number, and can become a super-composite when its value exceeds in value any values in its immediate neighborhood. Typically super-composites have \( f(N) > 1 \). It is our purpose here to develop some general formulas for quickly calculating \( f(N) \) for any positive integer.

Our starting point is to write down the following values for \( f(N) \) when \( N=p \) is a prime. These read-

\[
\begin{align*}
f(p) &= 0 \\
f(p^2) &= \frac{1}{p} \\
f(p^3) &= \frac{1 + p}{p^2} \\
f(p^4) &= \frac{1 + p + p^2}{p^3}
\end{align*}
\]

On generalizing this leads to the unique result-

\[
f(p^n) = \frac{1 + p + p^2 + \ldots + p^{n-2}}{p^{n-1}} = \frac{1}{(p - 1)} \left(1 - \frac{1}{p^{n-1}}\right)
\]

From this one sees at once that-

\[
f(343) = f(7^3) = \frac{(49-1)}{[6(49)]=8/49 \quad \text{and} \quad f(625) = f(5^4) = 31/125}
\]

Also the values of \( f(pq) \) and \( f(p^2q) \), where \( p \) and \( q \) are primes, may be written as-

\[
\begin{align*}
f(pq) &= \frac{p + q}{pq} \quad \text{and} \quad f(p^2q) = \frac{p + q + p(p + q)}{p^2q}
\end{align*}
\]

Other products involving primes \( p, q, r \) etc may be derived by using the basic definition of \( f(N) \).

We have, for instance, that -

\[
f(pqr) = \frac{p+q+r+(pr+qr+qp)}{pqr}
\]
With the above general forms for \( f(N) \), where \( N \) involve only products of primes, we are now in a position to quickly calculate the values of the number fraction \( f(N) \) for any \( N \). One knows from number theory that any positive integer \( m \) may be written as-

\[
N = \prod_{k=1}^{m} (p_k)^{a_k}
\]

where \( p_k \) is the \( k \)th prime and \( a_k \) a positive integer power. So if, for example, we ask what is \( f(117) \), we can write-

\[
f(117) = f(3^2 \cdot 13) = [(3+13) + 3(3+13)]/117 = \frac{64}{117}
\]

We could of course also work out this value using the identity-

\[
\sigma(N) = Nf(N) + N + 1
\]

Many computer programs such as MAPLE and MATHEMATICA already have built in values for the sigma function. For the above case \( \sigma(117) = 182 \) so that–

\[
f(117) = (182-117-1)/117 = \frac{64}{117}
\]

The big advantage of \( f(N) \) over \( \sigma(N) \) is that \( f(N) = 0 \) directly predicts the location of primes, and that it is also able to quickly predict super-composites.

Consider looking in the neighborhood of the number –

\[
N = 2^53^25^2 = 21600
\]

Here \( \sigma(N) = 78120 \) and \( f(N) = 2.61662 \). The latter indicates that 21600 is a super-composite. Drawing a graph of \( f(N) \) in a twenty range interval about \( N = 21600 \) yields-
The presence of a super-composite at $N=21600$ is confirmed. Also one has the bonus result that there are a pair of twin primes in its immediate neighborhood. From earlier results we know that all primes greater than three have the form $6n\pm1$, so we can expect other primes to possibly be found at distances $6k$ from the primes 21599 and 21601, where $k=\pm1,\pm2,\pm3$, etc. We find primes at $21617=21599+18$, $21613=21601+12$, $21611=21599+12$, $21589=21601-12$, and $21587=21599-12$. So indeed they are separated by factors of six from one or the other of the twin primes shown. Another interesting observation is that there appears to be a weak reflective symmetry about the super-composite for which we have no explanation.

The finding of twin primes in the above example about 21600 is no coincidence. If one looks at our well known hexagonal integer spiral shown-
it becomes clear that primes above three have the form $6n \pm 1$ and all twin primes are possible only when $N$ is a multiple of six. In the graph we show twin primes for $n=6,12,18,30$. The gaps in the primes at 25 and 35 and the gap in twin primes corresponding to $6n=24$ are allowed because the above conditions are necessary but not sufficient.

Let us see if we can find another of these primes. Take $N=6(56545) = 279270$ which lies along the $6n$ axis in the above diagram. There $f(279270) = 1.539161976$ and both $N+1$ and $N-1$ are primes constituting a double prime. Another example is $N=219252$ which has the twin primes 219253 and 219251. Note, however, that $N=6n$ does not necessarily imply that one or both numbers $N \pm 1$ are prime. For example, $N=24=6(4)$ in the above graph leads to one prime at 23 but a composite at 25.

If we take a larger number at random such as –

$$N=14829=6(271)+3$$

we know it lies along the line $6n+3$ separated from the two prime lines at $6n+1$ and $6n+5$. So we suspect that –

$$N-2=14829 \quad \text{and} \quad N+2=14831$$

might be primes. A computer run shows that $f(14829)$ and $f(14831)$ are both zero so that these numbers are indeed primes. Other prime possibilities would be $14829 \pm 6k$ and $14831 \pm 6k$. Since the smallest prime along line $6n-1$ is five and the smallest prime along line $6n+1$ is seven, it is also true that all primes greater than three have the form–
Any of these primes will have their number fraction vanish.

A last point concerning the $f(N)$ function is its value when $N$ is a semi-prime $N=pq$, where $p$ and $q$ are primes. Such semi-primes are of importance in connection with public key cryptography. They are notoriously difficult to factor when $N$ gets large. Take the large semi-prime-

$$N=320512741=6(53418790)+1$$

It lies along the prime line $6n+1$ and has $f(N)=0.0001136990682$. The very small (but non-vanishing) value of $f(N)$ is characteristic for such semi-primes. Any semi-prime will always lie along either the $6n+1$ or $6n-1$ prime lines. From this fact we can see what the form of $p$ and $q$ will look like. For the above example $p=6n\pm1$ and $q=6m\pm1$, with the sign the same in both expressions. Another semi-prime is-

$$N=3285899=6(547650)-1$$

Which lies along the radial line $6n-1$ and has $f(N)=0.00113850426$. Here we know that $p=6n+1$ and $q=6m-1$. Again $f(N)$ has a value just slightly above zero. To factor this number we write-

$$ (6n+1)(6m-1)=N \quad \text{or the equivalent} \quad 6x-y=(N+1)/6$$

On setting $nm=x$ and $n-m=y$ and letting $b$ be the nearest integer to $(N+1)/36$, we get-

$$x=nm=b+k \quad \text{and} \quad y=n-m=6k \quad \text{where} \quad k \quad \text{is an integer to be determined}$$

Solving for $n$ one gets-

$$n = 3k + \sqrt{9k^2 + b + k}$$

Since $k$ and $b$ must be integers one requires that the radical yields a real positive integer. For the special case above where $b=91275$, we get the radical equals 305 at $k=-14$. Hence we have $n=263$. This yields $p=1579$ and $q=2081$ for the prime number components of the semi-prime $N=3285899$.

September 1, 2016