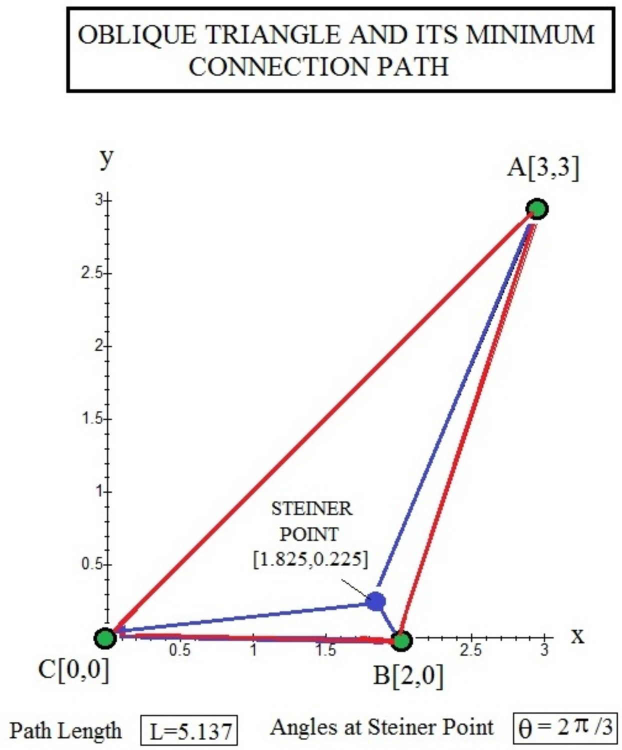
SHORTEST ROUTE CONNECTING N POINTS IN A PLANE

The shortest distance between point A[x_A,0] and point B[x_B,0] along the x-axis is a straight line of length L=x_B-x_A. If one adds a third point C[x_C,y_C] , not located on the x-axis , the minimum path connecting all three becomes more complicated. In that case the shortest path will be three straight lines intersecting at a fourth point known as the Steiner Point (named after the mathematician Jakob Steiner(1796-1865)). A typical minimum path where the three points A[3,3], B[2,0], and C [0,0] form the vertexes of an oblique looks as follows-



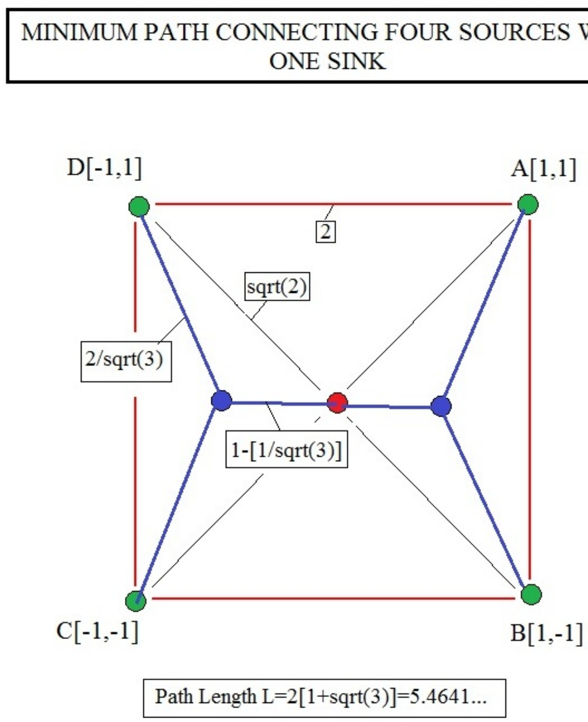
We see there that the minimum path connecting all three vertexes is the three line configuration shown in blue. They intersect at a common point located at [1.825,0.225]. The point isreferred to as a Steiner point. What is interesting about such a three prong path is that the intersection produces three identical angles of θ=2π/3 rad=120 degree each between neighboring lines. The Steiner Point is easiest to locate by performing a contour-plot of the path length L and then picking the lowest value contour. For the present case L=5.137. Using the Law of cosines one also has-

$$\cos(\theta) = -\frac{5.536}{11.083} = -0.499 \qquad so \quad that \quad \theta = 119.96 \deg ree$$

This is within the error range of the correct value of 120 degree. We can refer to the tree-prong configuration inside any triangle as a Steiner Fork which has the lines entering the Steiner Point at 120 degrees but whose three prongs may be of unequal length.

It is our purpose here to generalize the Steiner results by determining the minimum path length connecting the vertexes of any regular polygon with the polygon center. We will refer to the vertexes as source points and the central point as the sink. We will show how to calculate the optimum trajectory connecting the sources with the sink by a technique of breaking up the polygon into sub-triangles and then placing a Steiner Fork inside some of the triangles.

Let us begin our examination with a square having sources at its four vertexes located at A[1,1], B[1,-1], C[-1,-1] and D[-1,1]. By drawing in the two diagonal lines from opposite vertexes we get four isosceles triangles as shown-

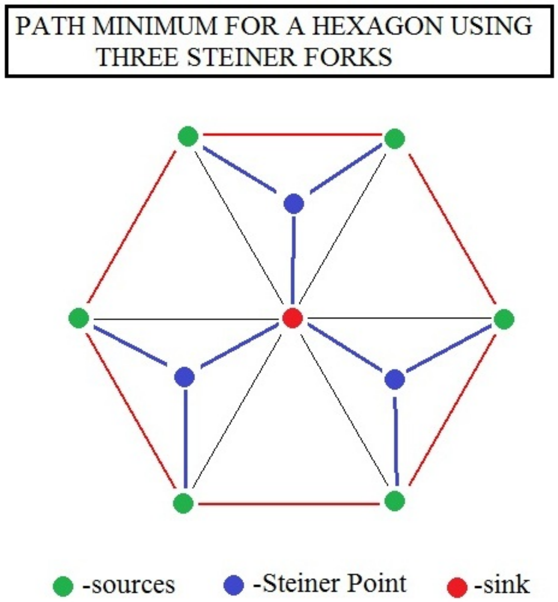


Placing a Steiner fork into two of the opposing triangles we get the blue line path yielding the minimum path length L connecting all four vertexes with the sink at the center of the square. A little trigonometry to calculate the length of the path's increment between the two Steiner Points(blue circles) yields 2[1-1/sqrt(3)]. Thus the total length of the optimum path becomes-

$$L=2[1+\sqrt{3}]=5.4641..$$

If we now look at an alternate path of four radial lines running from the vertexes to the center sink we find the slightly larger value of L=4sqrt(2)=5.6568… This is larger than the optimum path by some two percent. The super-positioning multiple Steiner Forks seems to be working.

We next proceed on to the case of six source points at the vertexes of a regular hexagon connected with a single sink at the polygon center as shown-

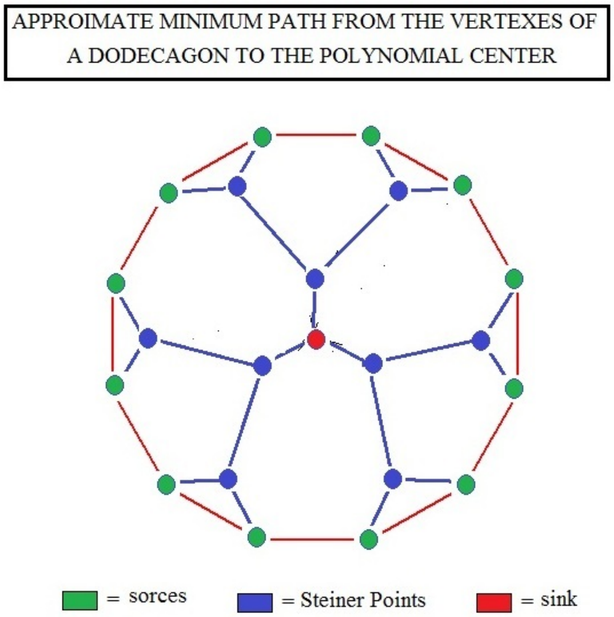


We first draw in six equilateral triangles and then place Steiner Forks into three of them skipping every second triangle. The Steiner points for such a hexagon lie at the geometric center of the triangles. With the 120 degree angle between the legs being obvious. There are a total of nine equal length segments with each segment given by l=w/sqrt(3), where w is side-length between neighboring vertexes. Thus the optimum path will equal-

$$L = \frac{9w}{2\cos(30)} = (3\sqrt{3})w = (5.19615..)w$$

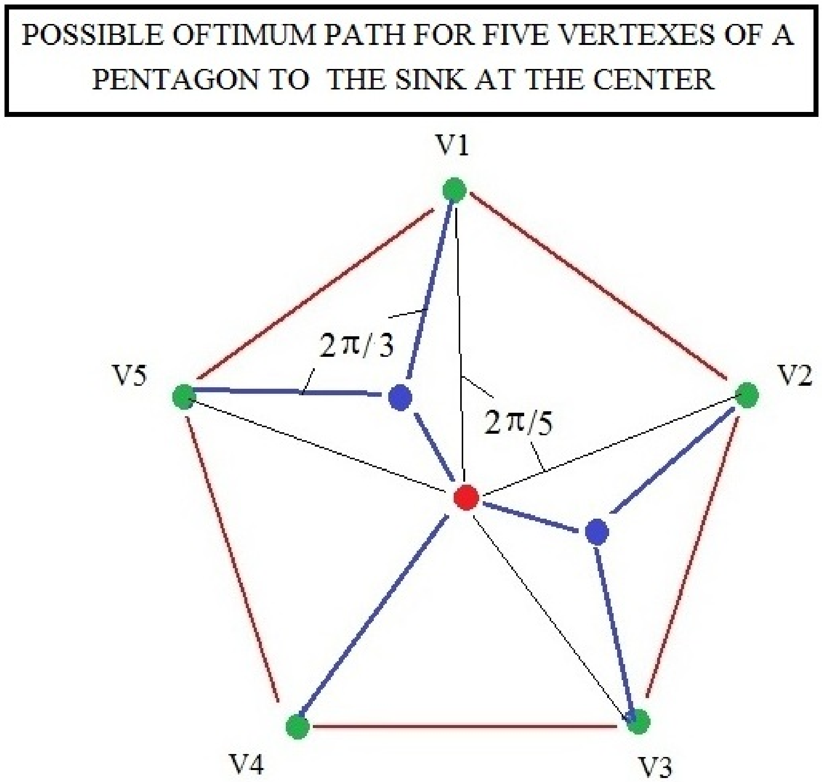
We can compare this number with that of a non-optimum path consisting of six radial lines emanating from each source and intersecting the sink at the polygon center. The path length will be L=6w and so some 15 percent longer.

We continue on with twelve sources placed at the vertexes of a regular dodecagon and one sink at the polygon center. First we draw in twelve equal isosceles triangles. These triangles appear to be too long suggesting we place the first six Steiner Forks just inside the polygon perimeter and leaving the third leg slightly adjustable from the 120 degree restriction of a standard Steiner Fork. We next place an equilateral triangle centered on the origin and containing a single Steiner fork with equal length legs. The end of these legs are then connected by straight lines to the incoming third legs of the outer forks. The resultant configuration looks like this-



The pattern is reminiscent of soap bubble configurations. This should not be surprising since bubbles are minimum energy surfaces and thus are consistent with the type of minimum trajectories we are discussing here. Since this time the geometry did not allow us to use exact Steiner forks for some of the forks and the fact that the size of the inner equilateral triangle can take on different values, we cannot be sure that the blue line trajectory shown in the figure is indeed a minimum path. One suspects however that the result must lie very close to an optimum. A ruler measurement finds the length of the near optimum path is L=15.8w compared to L=23.0w for a path consisting of twelve radial lines. Here gain w is the straight line distance between neighboring vertexes.

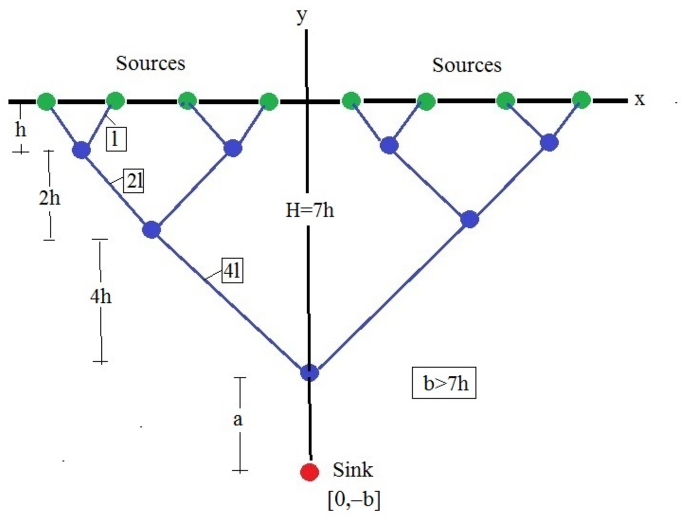
Optimum paths where the number of polygon vertexes are odd are more difficult to find since the pattern symmetry is destroyed. For example, taking a five vertex polygon configuration a possible pathe looks like this-



Here we have two Steiner Forks plus one direct radial line. It is likely to be an optimum but we have cannot be sure. By Changing the radial line from V4 to the center to a line from V4 to V3 will not change the total path length so it is also a possible minimum.

In addition to having the sources lie on the periphery of regular polygons, it is also possible to have 2n equally spaced sources along the x axis and then determine a minimum path connecting these to a single point (sink) along the negative y axis. For the case of eight points along the x-axis we get the following picture-

SOURCE-SINK PATH USING SEVEN TRUNCATED STEINER FORKS



Here the spacing between the sources along the x-axis is taken as w and the height h and slant line length l . of the first set of Steiner Forks are-

$$h = \frac{w}{2\tan(\pi/3)} = \frac{w}{2\sqrt{3}} \qquad and \qquad l = \frac{w}{2\sin(\pi/3)} = \frac{w}{\sqrt{3}}$$

We have cut the lower leg of the Steiner Forks so that the paths all meet at [x,y]=[0,-7h] . There are a total of 4+2+1=7 Steiner Forks whose slant lengths for the upper two branches are l, 2l, and 4l , respectively. This yields a total path length from the sources to the collection sink of-

L=b-7w/2sqrt(3) +24w/sqrt(3)=b+41w/[(2sqrt(3)]

, where [0,-b] represents the coordinate of the sink and b>7h. If we change the angle between the upper two branches of any of the truncated Steiner Points from 2π/3 radians to θ, the new length is–

$$L = b + w\left\{\frac{12}{\sin(\theta/2)} - \frac{7}{2\tan(\theta/2)}\right\}$$

On plotting β=(L-b)/w versus θ we find a minimum of β =11.45 at θ=2.55 radians=146.10 degrees . Thus the 120 degree angle needs to be replaced by a 146.10 degree angle to minimize the length L for the present eight source configuration.

An interesting observation about this last configuration is that, if water lines are connected from the sink to the eight points along the x-axis , the water flow from each nozzle would be identical since the path length are equal. Such a configuration might be of interest in connection with spray painting or agricultural irrigation.

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