GENERATING ARCTANGENT FORMULAS FOR PI

Starting in about 1700 through the 1970s, the preferred way for finding additional digits of Pi was the use of arctangent formulas. Although this approach has now been made obsolete by the numerical evaluation of elliptic integrals using AGM (algebraic-geometric mean) techniques, it is nevertheless of interest to see if improved forms for arctangent formulas can be discovered. We give you here a quick review of how these formulas (of which there are an infinite number) are generated. Start with the function-

$$F = \ln \prod_{n=1}^{K} [(N_n + i)^{p_n}] = \sum_{n=1}^{K} [p_n \ln(N_n + i)]$$

and let –

$$G = \prod_{n=1}^{K} (N_n + i)^{p_n} = (N_1 + i)^{p_1} (N_2 + i)^{p_2} \cdots (N_K + i)^{p_K}$$

be the complex polynomial whose log is defined by F. If one now takes the imaginary part of both definitions for F, we find-

$$\arctan\left[\frac{\mathrm{Im}(G)}{\mathrm{Re}(G)}\right] = \sum_{n=1}^{K} p_n \arctan\left[\frac{1}{N_n}\right]$$

Next, by demanding that Im(G)=Re(G), one obtains the general K term arctan formula-

$$\frac{\pi}{4} = p_1 \arctan\left[\frac{1}{N_1}\right] + p_2 \arctan\left[\frac{1}{N_2}\right] + \cdots + p_K \arctan\left[\frac{1}{N_K}\right]$$

By adjusting the ps, Ns, and K to different integer values one can theoretically find an infinite number of arctangent formulas from this last equality. The difficulty in actually finding these formulas arises through the requirement that Re(G)=Im(G). Typically one looks for solutions for which the integer N_n is large and the value of K remains small if rapid convergence is desired. I give you here a list of the historically better known arctan formulas-

Gregory(1670): $p_1 = 1$, $N_1 = 1$, K=1

Machin(1706): $p_1 = 4$, $N_1 = 5$, $p_2 = -1$, $N_2 = 239$, K=2

Euler(1748): $p_1 = 5$, $N_1 = 7$, $p_2 = 2$, $N_2 = 79/3$, K=2

Gauss(1840): $p_1 = 12$, $N_1 = 18$, $p_2 = 8$, $N_2 = 57$, $p_3 = -5$, $N_3 = 239$, K=3

Stoermer(1896): $p_1 = 6$, $N_1 = 8$, $p_2 = 2$, $N_2 = 57$, $p_3 = 1$, $N_3 = 239$, K=3

There are many additional multiple term arctan formulas found in the literature. They are usually generated by multiple applications of the arctan identities-

$$\arctan\left[\frac{1}{N}\right] = \arctan\left[\frac{1}{(N+1)}\right] + \arctan\left[\frac{1}{(N^2+N+1)}\right]$$

and

$$\arctan\left[\frac{1}{N}\right] = 2\arctan\left[\frac{1}{2N}\right] - \arctan\left[\frac{1}{N(4N^2+3)}\right]$$

both of which follow from the above formula for G, and are ones which guarantee numerators of unity in the arctan components. I remember playing with these formulas about twenty years ago (1987) trying to find an arctangent formula for Pi which would converge rapidly and thus contain only large values of N_n. After some effort, I came up with the following four term formula-

$$\frac{\pi}{4} = 12\arctan\left[\frac{1}{38}\right] + 20\arctan\left[\frac{1}{57}\right] + 7\arctan\left[\frac{1}{239}\right] + 24\arctan\left[\frac{1}{268}\right]$$

It converges quite rapidly, requiring just 31 terms in the standard arctangent series to yield the hundred digit accurate result-

Pi=
3.14159265358979323846264338327950288419716939937510582097494459230781640628620899862803482534211706 8

That this formula is correct is easily verified by noting that the real and imaginary parts of the number $\mathbf{(38+i)^{12}(57+i)^{20}(239+i)^{7}(268+i)^{24}}$, although quite large, are equal. The quickest way to derive this four term arctan formula is to apply the identity-

$$\arctan[\frac{1}{N}] = 8\arctan[\frac{1}{8N}] - 4\arctan[\frac{1}{4N(64N^2+3)}] - 2\arctan[\frac{1}{2N(16N^2+3)}] - \arctan[\frac{1}{N(4N^2+3)}]$$

for N=1 plus the Stoermer result and the fact that arctan(1/7)=arctan(1/8)+arctan(1/57).

On the internet one can find many programs which evaluate functions up to million place accuracy. One of these can be found at-

http://www.alpertron.com.ar/BIGCALC.HTM

We have used this calculator to evaluate the above four term arctan formula for π to 1000 places. Here is the result-

π=3.141 592 653 589 793 238 462 643 383 279 502 884 197 169 399 375 105 820 974 944 592
307 816 406 286 208 998 628 034 825 342 117 067 982 148 086 513 282 306 647 093 844 609
550 582 231 725 359 408 128 481 117 450 284 102 701 938 521 105 559 644 622 948 954 930
381 964 428 810 975 665 933 446 128 475 648 233 786 783 165 271 201 909 145 648 566 923
460 348 610 454 326 648 213 393 607 260 249 141 273 724 587 006 606 315 588 174 881 520
920 962 829 254 091 715 364 367 892 590 360 011 330 530 548 820 466 521 384 146 951 941
511 609 433 057 270 365 759 591 953 092 186 117 381 932 611 793 105 118 548 074 462 379
962 749 567 351 885 752 724 891 227 938 183 011 949 129 833 673 362 440 656 643 086 021
394 946 395 224 737 190 702 179 860 943 702 770 539 217 176 293 176 752 384 674 818 467
669 405 132 000 568 127 145 263 560 827 785 771 342 757 789 609 173 637 178 721 468 440
901 224 953 430 146 549 585 371 050 792 279 689 258 923 542 019 956 112 129 021 960 864
034 418 159 813 629 774 771 309 960 518 707 211 349 999 998 372 978 049 951 059 731 732
816 096 318 595 024 459 455 346 908 302 642 522 308 253 344 685 035 261 931 188 171 010
003 137 838 752 886 587 533 208 381 420 617 177 669 147 303 598 253 490 428 755 468 731
159 562 863 882 353 787 593 751 957 781 857 780 532 171 226 806 613 001 927 876 611 195
909 216 420 199

This result agrees exactly with the first 999 digits of the 10,000 place table found in the appendix of an excellent paperback book on π by Petr Bergmann, "A History of Pi".

If one is willing to dispense with the presence of one in the numerator terms of the above arctan formula, it becomes possible to write our four term arctan formula as-

$$\pi = 732\arctan(\frac{1}{268}) + 304\arctan(\frac{25}{19157}) + 128\arctan(\frac{903}{133863721}) + 28\arctan(\frac{29}{64053})$$

We can use this formula to match the Dutch mathematician Ludolph van Ceulen's 35 place accurate result for π (which took him a lifetime to calculate back in 1615 using the Archimedes approach) by truncating the arctan series after just a few terms. It takes only a split second on our PC to carry out the operation-

$$\pi \cong 732\sum_{n=0}^{7}\frac{(-1)^n}{(2n+1)(268)^{2n+1}} + 304\sum_{n=0}^{6}\frac{(-1)^n}{(2n+1)(19157/25)^{2n+1}} +$$

$$128\sum_{n=0}^{4}\frac{(-1)^n}{(2n+1)(133863721/903)^{2n+1}} + 28\sum_{n=0}^{5}\frac{(-1)^n}{(2n+1)(64053/29)^{2n+1}}$$

to yield the 36 place accurate result-

$$\pi \approx 3.141592653589793238462643383279502884$$

In parts of Europe one still refers to Pi as the Ludolph Number in his honor.

One can also expand the standard Machin formula to yield the five term arctan result-

$$\pi = 128\arctan[\frac{1}{40}] - 4\arctan\left[\frac{1}{239}\right] - 16\arctan\left[\frac{1}{515}\right] - 32\arctan\left[\frac{1}{4030}\right] - 64\arctan\left[\frac{1}{32060}\right]$$

or the six term formula-

$$\pi = 256\arctan[\frac{1}{80}] - 4\arctan[\frac{1}{239}] - 16\arctan[\frac{1}{515}] - 32\arctan[\frac{1}{4030}]$$
$$- 64\arctan[\frac{1}{32060}] - 128\arctan[\frac{1}{256120}]$$

and even the seven term arctan formula-

$$\pi = 512\arctan[\frac{1}{160}] - 4\arctan[\frac{1}{239}] - 16\arctan[\frac{1}{515}] - 32\arctan[\frac{1}{4030}]$$
$$- 64\arctan[\frac{1}{32060}] - 128\arctan[\frac{1}{256120}] - 256\arctan[\frac{1}{2048240}]$$

Note that, unlike with AGM methods, the evaluation of Pi by arctan formulas requires no taking of roots of large numbers and only requires an accurate evaluation of the quotients-

$$Q_n(N) = (-1)^n / [(2n+1)N^{2n+1}]$$

with the convergence rate improving with ever larger values of N.