GENERATING ARCTANGENT FORMULAS FOR PI

Starting in about 1700 through the 1970s, the preferred way for finding additional digits of Pi was the use of arctangent formulas. Although this approach has now been made obsolete by the numerical evaluation of elliptic integrals using AGM (algebraic-geometric mean) techniques, it is nevertheless of interest to see if improved forms for arctangent formulas can be discovered. We give you here a quick review of how these formulas (of which there are an infinite number) are generated. Start with the function-

\[ F = \ln \prod_{n=1}^{K} [(N_n + i)^{p_n}] = \sum_{n=1}^{K} [p_n \ln(N_n + i)] \]

and let -

\[ G = \prod_{n=1}^{K} (N_n + i)^{p_n} = (N_1 + i)^{p_1} (N_2 + i)^{p_2} \cdots (N_K + i)^{p_K} \]

be the complex polynomial whose log is defined by F. If one now takes the imaginary part of both definitions for F, we find-

\[ \arctan\left[ \frac{\text{Im}(G)}{\text{Re}(G)} \right] = \sum_{n=1}^{K} p_n \arctan\left[ \frac{1}{N_n} \right] \]

Next, by demanding that \( \text{Im}(G)=\text{Re}(G) \), one obtains the general K term arctan formula-

\[ \frac{\pi}{4} = p_1 \arctan\left[ \frac{1}{N_1} \right] + p_2 \arctan\left[ \frac{1}{N_2} \right] + \cdots + p_K \arctan\left[ \frac{1}{N_K} \right] \]

By adjusting the ps, Ns, and K to different integer values one can theoretically find an infinite number of arctangent formulas from this last equality. The difficulty in actually finding these formulas arises through the requirement that \( \text{Re}(G)=\text{Im}(G) \). Typically one looks for solutions for which the integer \( N_n \) is large and the value of K remains small if rapid convergence is desired. I give you here a list of the historically better known arctan formulas-

- Gregory(1670): \( p_1 = 1, N_1 = 1, K=1 \)
- Machin(1706): \( p_1 = 4, N_1 = 5, p_2 = -1, N_2 = 239, K=2 \)
- Euler(1748): \( p_1 = 5, N_1 = 7, p_2 = 2, N_2 = 79/3, K=2 \)
Gauss (1840): \( p_1 = 12, \ N_1 = 18, \ p_2 = 8, \ N_2 = 57, \ p_3 = -5, \ N_3 = 239, \ K = 3 \)

Stoermer (1896): \( p_1 = 6, \ N_1 = 8, \ p_2 = 2, \ N_2 = 57, \ p_3 = 1, \ N_3 = 239, \ K = 3 \)

There are many additional multiple term arctan formulas found in the literature. They are usually generated by multiple applications of the arctan identities:

\[
\arctan \left( \frac{1}{N} \right) = \arctan \left( \frac{1}{N+1} \right) + \arctan \left( \frac{1}{N^2 + N + 1} \right)
\]

and

\[
\arctan \left( \frac{1}{N} \right) = 2 \arctan \left( \frac{1}{2N} \right) - \arctan \left( \frac{1}{N(4N^2 + 3)} \right)
\]

both of which follow from the above formula for G, and are ones which guarantee numerators of unity in the arctan components. I remember playing with these formulas about twenty years ago (1987) trying to find an arctangent formula for \( \pi \) which would converge rapidly and thus contain only large values of \( N_n \). After some effort, I came up with the following four term formula-

\[
\frac{\pi}{4} = 12 \arctan \left( \frac{1}{38} \right) + 20 \arctan \left( \frac{1}{57} \right) + 7 \arctan \left( \frac{1}{239} \right) + 24 \arctan \left( \frac{1}{268} \right)
\]

It converges quite rapidly, requiring just 31 terms in the standard arctangent series to yield the hundred digit accurate result-

\[
\pi = 3.141592653589793238462643383279502884197169399375105820974944592307816406286208998628034825342117068
\]

That this formula is correct is easily verified by noting that the real and imaginary parts of the number \( (38 + i)^{12}(57 + i)^{20}(239 + i)^{7}(268 + i)^{24} \), although quite large, are equal. The quickest way to derive this four term arctan formula is to apply the identity-

\[
\arctan \left( \frac{1}{N} \right) = 8 \arctan \left( \frac{1}{8N} \right) - 4 \arctan \left( \frac{1}{4N(64N^2 + 3)} \right) - 2 \arctan \left( \frac{1}{2N(16N^2 + 3)} \right) - \arctan \left( \frac{1}{N(4N^2 + 3)} \right)
\]

for \( N = 1 \) plus the Stoermer result and the fact that \( \arctan(1/7) = \arctan(1/8) + \arctan(1/57) \).
On the internet one can find many programs which evaluate functions up to million place accuracy. One of these can be found at-

http://www.alpertron.com.ar/BIGCALC.HTM

We have used this calculator to evaluate the above four term arctan formula for $\pi$ to 1000 places. Here is the result-

$$\pi = 3.14159265358979323846264338327950288419716939937510582097494459230781640628620899862803482534228602498214053048211470349482102701838452110555964462294889549303819644288109756659334461284756482337867831652712019091451464856692346034861045432664821339360726024914127372458700660631558817488152092096282925409171536436789259036021131033053045828204665213841469519415116094330572703657595919530921861173819326117931051185480744623796274956735188575272489122793818301194912983367336244065664308602139494639522473719071986094370277053921762917675238467481846766940513200056812714526356082778577134275778960917363717872146844090122495343014654958537105079227969925892354201995611212902196086403441851981362977477130996051870721134999998372978049951059731732816096318595024459455346908302642522308253344685035261931187101003137838752886587533208381420617177669147303598253490428755468731159562863882353787593751957781857780532171226806613001927876611195909216420199

This result agrees exactly with the first 999 digits of the 10,000 place table found in the appendix of an excellent paperback book on $\pi$ by Petr Bergmann, “A History of Pi”.

If one is willing to dispense with the presence of one in the numerator terms of the above arctan formula, it becomes possible to write our four term arctan formula as-

$$\pi = 732 \arctan\left(\frac{1}{268}\right) + 304 \arctan\left(\frac{25}{19157}\right) + 128 \arctan\left(\frac{903}{133863721}\right) + 28 \arctan\left(\frac{29}{64053}\right)$$

We can use this formula to match the Dutch mathematician Ludolph van Ceulen’s 35 place accurate result for $\pi$ (which took him a lifetime to calculate back in 1615 using the Archimedes approach) by truncating the arctan series after just a few terms. It takes only a split second on our PC to carry out the operation-

$$\pi \approx 732 \sum_{n=0}^{7} \frac{(-1)^n}{(2n+1)(268)^{2n+1}} + 304 \sum_{n=0}^{6} \frac{(-1)^n}{(2n+1)(19157/25)^{2n+1}} + 128 \sum_{n=0}^{4} \frac{(-1)^n}{(2n+1)(133863721/903)^{2n+1}} + 28 \sum_{n=0}^{5} \frac{(-1)^n}{(2n+1)(64053/29)^{2n+1}}$$
to yield the 36 place accurate result-

\[ \pi \approx 3.141592653589793238462643383279502884 \]

In parts of Europe one still refers to Pi as the Ludolph Number in his honor.

One can also expand the standard Machin formula to yield the five term arctan result-

\[
\pi = 128 \arctan \left( \frac{1}{80} \right) - 4 \arctan \left( \frac{1}{239} \right) - 16 \arctan \left( \frac{1}{515} \right) - 32 \arctan \left( \frac{1}{4030} \right) - 64 \arctan \left( \frac{1}{32060} \right)
\]

or the six term formula-

\[
\pi = 256 \arctan \left( \frac{1}{80} \right) - 4 \arctan \left( \frac{1}{239} \right) - 16 \arctan \left( \frac{1}{515} \right) - 32 \arctan \left( \frac{1}{4030} \right) - 64 \arctan \left( \frac{1}{32060} \right) - 128 \arctan \left( \frac{1}{256120} \right)
\]

and even the seven term arctan formula-

\[
\pi = 512 \arctan \left( \frac{1}{160} \right) - 4 \arctan \left( \frac{1}{239} \right) - 16 \arctan \left( \frac{1}{515} \right) - 32 \arctan \left( \frac{1}{4030} \right) - 64 \arctan \left( \frac{1}{32060} \right) - 128 \arctan \left( \frac{1}{256120} \right) - 256 \arctan \left( \frac{1}{2048240} \right)
\]

Note that, unlike with AGM methods, the evaluation of Pi by arctan formulas requires no taking of roots of large numbers and only requires an accurate evaluation of the quotients-

\[ Q_n(N) = (-1)^a / [(2n + 1)N^{2n+1}] \]

with the convergence rate improving with ever larger values of N.