MATHEMATICS OF THE PLANCK AND STEFAN-BOLTZMANN RADIATION LAWS FOR BLACK BODIES

In a famous 1900 scientific publication Max Planck for the first time published the correct blackbody radiation law valid for all frequencies \( \nu \). It reads-

\[
U_\nu (T) = \frac{8\pi h \nu^3}{c^3} \left[ \frac{1}{\exp(h \nu/kT) - 1} \right]
\]

Here \( U_\nu(T) \) is the spectral radiance, \( c \) the speed of light, \( h \) is Planck’s constant, \( k \) the Boltzmann constant and \( T \) the absolute temperature of the radiating body. When integrated over the full frequency range it leads to the earlier known Stefan-Boltzmann law-

\[
E = \sigma T^4 \quad \text{with} \quad \sigma = 5.670 \times 10^{-8} \, \text{J/s m}^2
\]

We work out here the mathematics behind these laws. Our starting point is to envisage standing electromagnetic waves in a cubical box of sides \( L \). The solution of the basic wave equation for standing electromagnetic waves in this box is-

\[
\psi(x,y,z,t) = \text{Const.} \sin(\omega t) \sin(n_x \pi x / L) \sin(n_y \pi y / L) \sin(n_z \pi z / L)
\]

where \( n_x, n_y, n_z \) have values of all the positive integers. The conditions that \( \psi(x,y,z,t) \) be zero on all six walls of the box, requires that the square of the angular frequency-

\[
\omega^2 = (2\pi \nu)^2 = \left( \frac{\pi \nu}{L} \right)^2 \cdot [n_x^2 + n_y^2 + n_z^2]
\]

This last result represents an octant of a sphere in phase space given by-

\[
[n_x^2 + n_y^2 + n_z^2] = \left[ \frac{2\nu L}{c} \right]^2 = R^2
\]

The number of modes within a shell over one octant of the sphere and of thickness \( dv \) will then be-
\[ N(\nu) d\nu = \frac{4\pi R^2 dR}{8} \cdot 2 = \frac{8\pi L^3}{c^3} \nu^2 d\nu \]

The factor two comes from the polarization of each mode. Next the energy radiated per unit volume and unit frequency will be-

\[ U_\nu(T) = \frac{N(\nu)}{L^3} E_\nu(T) \]

with \( E_\nu(T) \) the energy associated with each mode. In classical mechanics this energy value is just equal to \( kT \) and produces the Rayleigh-Jeans Law

\[ U_\nu(T) = \frac{8\pi kT}{c^3} \nu^2 \]

which unfortunately diverges at large frequencies \( \nu \). It was Plank’s great contribution to recognize that the energy of the various modes are quantized so that \( E = nh\nu \) where \( n=1,2,3,... \) He used the averaged value of his quantized \( E \) using Boltzmann statistics. This produces-

\[ \bar{E} = \sum_{n=1}^{\infty} \frac{nh\nu \exp(-nh\nu/kT)}{\exp(-nh\nu/kT)-1} = \frac{h\nu}{\exp(h\nu/kT)-1} \]

and the final result-

\[ U_\nu(T) d\nu = \frac{8\pi h\nu^3}{c^3} \left[ \frac{1}{\exp(h\nu/kT)-1} \right] d\nu \]

This is the well known Black Body Radiation Formula which is seen to vanish for both \( \nu=0 \) and \( \nu \) going to infinity. Recalling that \( c=\lambda \nu \), we can also write the Planck Radiation Law as radiant energy per wavelength. It reads-
We can plot these curves for several different values of temperature $T$. Rewriting things as

$$ F(\lambda, T) = \frac{V_\lambda(T)}{8\pi hc} = \frac{1}{\lambda^5} \left[ \frac{1}{\exp(\Delta / \lambda T) - 1} \right] $$

where

$$ \Delta = \frac{hc}{k} = \frac{(6.626 \times 10^{-34})(2.998 \times 10^8)}{1.3807 \times 10^{-23}} = 0.014387 \text{ m} \cdot \text{K} $$

We find the following plots for $T=4000\text{K}$, $5000\text{K}$, and $6000\text{K}$ in the wavelength range of $0<\lambda<2 \times 10^{-6} \text{ meters}$.

Notice how the total radiation increases rapidly with body temperature and that each curve at the maximum radiation point obeys the displacement formula $\lambda T = \text{Const}$. This
result is known in the literature as Wien’s Displacement Law. The constant has the value of 2.8976x10⁻³ mK.* It shows, for example, that the sun, whose surface temperature is 5800 K, has a maximum radiation at \( \lambda = 5000 \text{Å} = 0.5 \text{microns} \). To derive the result from the Plank radiation formula one notes that for the visible electromagnetic wavelength range the term \( \exp(h\nu/\lambda cT) \gg 1 \). Thus neglecting the term 1 in the denominator of the radiation formula and taking the derivative of \( V \) with respect to \( \lambda \), we find at the maximum of the curves that-

\[
\frac{dV(\lambda, T)}{d\lambda} = 0 = \left[ -\frac{5}{\lambda^5} + \frac{\Delta}{T\lambda^7} \right] \exp(-\Delta / \lambda T)
\]

That is -

\[
\lambda T = \frac{\Delta}{5} = 2.877 \times 10^{-3} \text{mK}
\]

This value is close to the value of the Wien Constant with the small discrepancy due to the neglect of 1 in denominator of the radiation formula.

On integrating the \( U_{\nu}(T) \) term over all frequencies, the resultant total radiation will be-

\[
J(T) = \int_{\nu=0}^{\infty} U_{\nu}(T) d\nu = \frac{8\pi k^4 T^4}{c^3 h^3} \int_{\nu=0}^{\infty} \frac{V^3}{[\exp(V) - 1]} = \frac{8\pi^5 k^4}{15 c^3 h^3} T^4
\]

As expected, the same result is found when integrating \( V_{\lambda}(T)d\lambda \). This last result is known as the Stefan-Boltzmann Law. It states that the total radiation from a black body goes as the 4th power of its absolute temperature.

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*The exact value of the Wien Constant is gotten by the differentiation-

\[
d\left\{ \frac{V_{\lambda}(T)\Delta^5}{8\pi c T^5} \right\} dz = d\left[ z^5 / (\exp(z) - 1) \right] = 0
\]

where \( z = \Delta / \lambda T \). Thus we need to evaluate –

\[
1 - \frac{z}{5} = \exp(-z)
\]

Starting with \( z_0 = 5 \) we can use the following iteration to find \( z_0 \).
\[ z_{n+1} = z_n + \left[ \frac{1 - (z_n / 5)}{(1/5) + \exp(-z_n)} \right] \]

This yields-

\[ z_0 = 5, \, z_1 = 4.9651\ldots, \, z_2 = 4.9651142317\ldots, \, \text{and} \, z_3 = 4.9651142317442763036987\ldots \]

Thus the correct expression for the Wein’s Displacement Law is-

\[ \lambda T = \frac{\Delta}{4.9651142} = 2.897617 \times 10^{-3} \]