COMPUTER CONSTRUCTION OF THE FIVE PLATONIC CONVEXPOLYHEDRA USING A GUIDING SPHERE

It is well known since the time of the ancient Greeks that there are just five solid convex polyhedra whose faces are in the form of regular polygons. They are the Tetrahedron, the Hexahedron, the Octahedron, the Dodecahedron, and the Icosahedron. Collectively they are referred to as the Platonic Solids. Euler showed quite early that the number of vertices V, the number of edges E, and the number of faces F of each of these polyhedra has is given by the formula-

$$V-E+F=2$$

Thus the simple cube (Hexahedron) has eight vertices and six faces. Therefore there must be 8+6-2=12 edges. An interesting property of these Platonic Solids is that a centered sphere will have all of the vertices of the polyhedron lie on the sphere's surface. This suggests that one can construct any of these Platonic Solids by connecting all vertex points on the sphere by straight lines . The regular spacing of the vertices on the sphere is determined by the number of faces of the Platonic Solid. The vertex placement is easiest to accomplish using a spherical coordinate system[r,θ,φ] and then converting the vertex location to Cartesian coordinates [x,y,z] via the transformations-

$$x=r\sin(\theta)\cos(\varphi),\quad y= r\sin(\theta)\sin(\varphi), \text{ and } \quad z=r\cos(\theta)$$

, where θ is the polar angle and φ the azimuthal angle. As a first demonstration, consider the Hexahedron and place its eight vertices at r=1 , $\varphi=\pm\pi/4$, $\pm 3\pi/4$ and equal to both $\theta_1=\pi/6$ and $\theta_2=5\pi/6$. Converting to Cartesian coordinates one has the eight vertices located at-

$$[x,y,z]=[\pm 1/2\,\mathrm{sqrt}(2), \pm 1/2\,\mathrm{sqrt}(2),\pm\mathrm{sqrt}(3)/2]$$

Note, as expected, the distance from the sphere center to any of these vertices is d=sqrt(1/8+1/8+3/4)=1. Connecting neighboring vertices we generate six squares forming the Hexahedron surface. The squares have side length s=2/sqrt(3) which is consistent with the length of 2 of the cube's diagonal. Using the MAPLE computer command-

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with(plots):
polyhedraplot([0,0,0], polytype=hexahedron, style=PATCH, axes=none,
scaling=CONSTRAINED);
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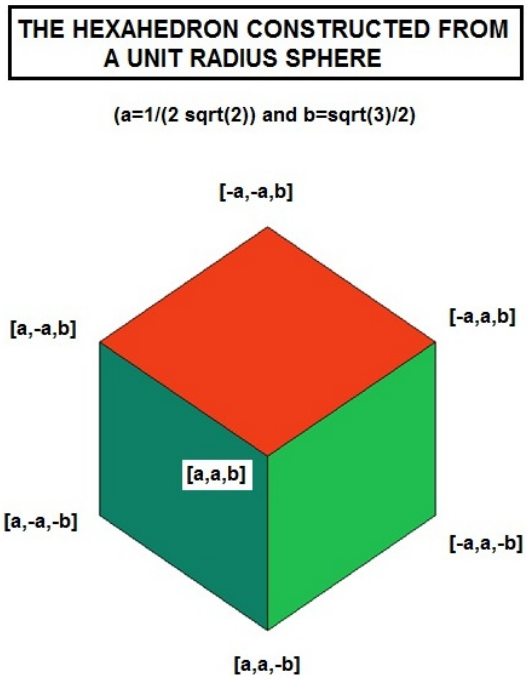
we obtain the Hexahedron graph-



Next, let us look at the Tetrahedron which is another of the Platonic solids. It has just four vertices and faces in the form of equilateral triangles. This time locating the vertices on the surface of a sphere of radius R is a bit more tricky. Clearly one vertex can be placed at the north pole of the sphere so that the remaining three are spaced at $\Delta\varphi$ intervals of $2\pi/3$. However the remaining three vertices do not lie on the 30deg south latitude as a cursory examination would suggest. Rather they are found at polar angle θ=arcos(-1/3) =121.634 deg. Converting to Cartesian coordinates, the vertices are found for a radius R sphere at-

[0, 0, R], [s/sqrt(3),0,-s sqrt(2/3)/4], [-s/2 sqrt(3)), s/2, -s sqrt(2/3)/4], [-s/(2sqrt(3),-s/2,-s sqrt(2/3)/4]

The distance from the sphere center to any of the vertices will be R=sqrt(3/2) s/2 and the sides of each of the equilateral triangles formed have length of exactly s=1. The area of any of the four faces of the Tetrahedron is found to be A=sqrt(3)/4. The height of the Tetrahedron is shown from the above given coordinates to be H=sqrt(2/3)). Hence the Tetrahedron volume will be-

$$V_{Tetra} = \frac{1}{3}\{\frac{\sqrt{3}}{4}\}\sqrt{\frac{2}{3}} = \frac{1}{6\sqrt{2}} \quad when \quad s = 1$$

Using our computer program-

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polyhedraplot([0,0,0], polytype=tetrahedron, style=PATCH,
scaling=CONSTRAINED, axes=none);
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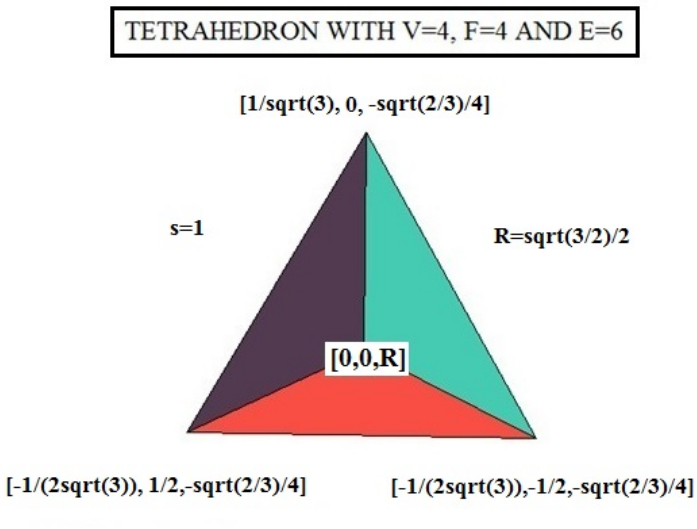
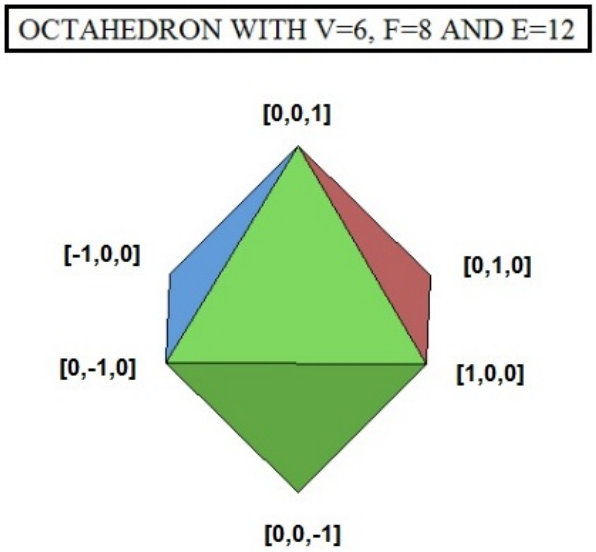
we obtain the figure-



Next we examine the Octahedron which has eight faces in the form of equilateral triangles. The six vertices on a unit radius sphere are easy to find by placing two of the vertices at the poles of the sphere and the four remaining vertices separated by $\pi/2$ radians from each other along the equator. In Cartesian coordinates the vertices lie at-

$$[0,0,\pm1],[\pm1,0,0]\ ,\ [0,\pm1,0]$$

Note again, the distance from the sphere center to each of the vertices of the Octahedron is just one. A graph of the Octahedron follows-



Each equilateral triangle has side length s=2sin(π/4)=sqrt(2) so that the total area of the solid is –

$$A=8\{\text{sqrt}(3)/4\}s^2=2\text{sqrt}(3)s^2=4\text{sqrt}(3)$$

To calculate the volume of a Octahedron we look at it as equivalent to eight triangular base pyramids of lateral side-length of 1 extending towards the sphere center and then use the pyramid formula-

$$V=8\{\text{ area of base triangle}\cdot\text{ pyramid height}\}/3=(8/3)\{\text{sqrt}(3)/2\cdot 1/\text{sqrt}(3)\}=\text{sqrt}(2)s^3/3=4/3$$

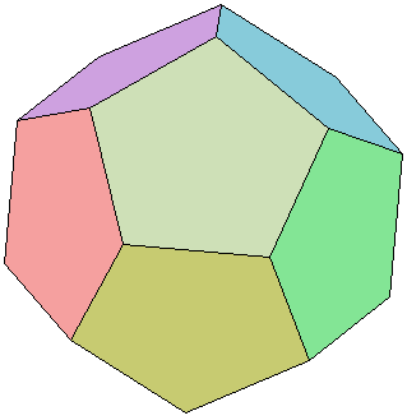
If the guiding sphere radius r is different from unity one needs to replace the side length s by rs in the above results.

The next Platonic Solid we examine is the Dodecahedron which consists of twelve pentagonal faces(F=12), a total of twenty vertices(V=20), and thirty edges(E=30). The Euler equation V-E+F=20-30+12=2 again holds. To construct this surface via computer we first need to locate the twenty vertices distributed at equal distance from each other about the sphere surface. To do so takes a bit more effort than in the earlier examples. This time point designations involve the Golden ratio f={1+sqrt(5)}/2. After some manipulations and choosing a sphere radius r=sqrt(3), one finds one possible distribution to be-

$$[\pm 1,\pm 1,\pm 1],\ [\pm f,0,\pm 1/f],\ [0,\pm 1/f,\pm f],\text{ and }[\pm 1/f,\pm f,0]$$

where [x,y,z] are the Cartesian coordinates of the twenty points. A computer plot yields-

DODECAHEDRON WITH V=20, F=12 AND E=30

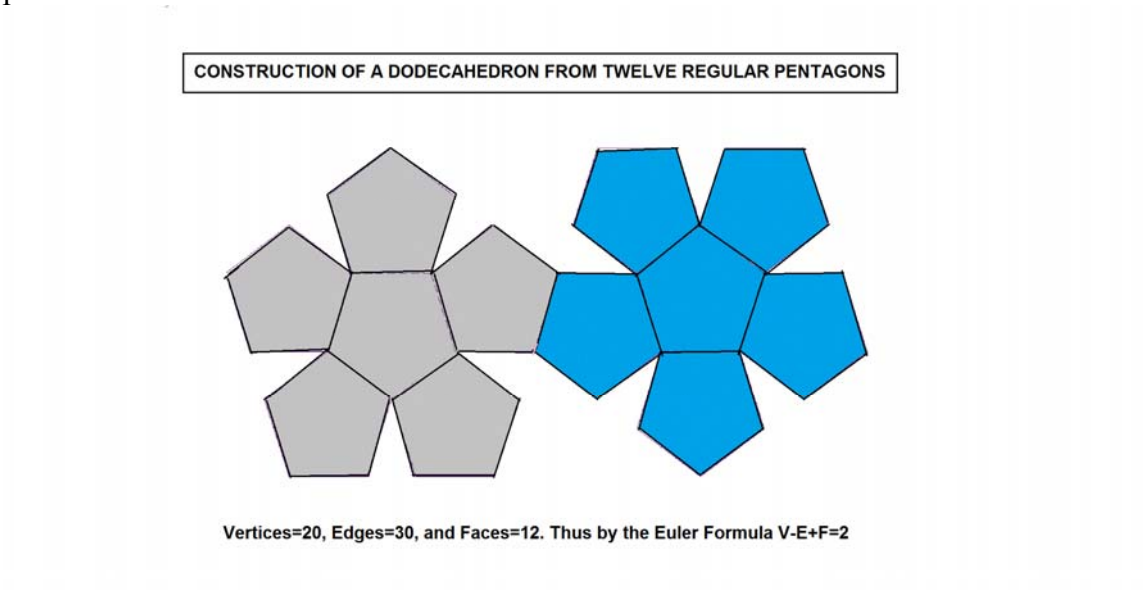


The surface area of each of the twelve pentagon faces of the Dodecahedron equals –

$$A_{pent}=(5/4)s^2/\tan(\pi/5)=\{(5/4)(1/\text{sqrt}(7-4f)\}s^2=1.720477\ s^2$$

so that the total surface area of a Dodecahedron is just twelve times this amount.

One also can construct models of the various Platonic Solids by the use of 2D cardboard cut-outs. Here is an example of such a cardboard pattern which when folded together produces a Dodecahedron –



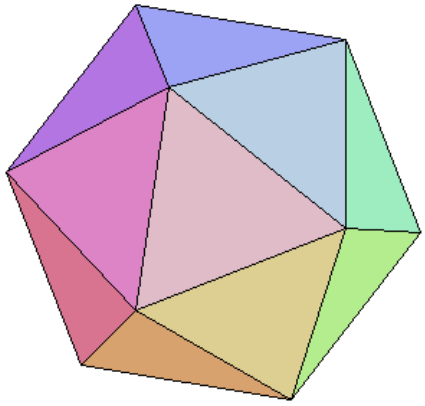
With plywood or wood veneer cladding , such models can be made quite rigid and last for years as demonstration tools in math classes. There is no limit to the size of such structures and thus they may find application for certain public art sculptures. Any one of these would be an improvement over the "Alachua (alias French Fries from Hell)" sculpture decorating our science library here at the University of Florida.

As the final of the five Platonic solids we look at the Icosahedron. It is composed of twenty equilateral faces (F=20), has just twelve vertices(V=12), and thirty edges (E=30). The [x,y,z] coordinates of equally spaced vertices on a sphere of radius r=sqrt($1+f^2$), where again f={1+sqrt(5)}/2, are as follows-

$$[0,\pm1,\pm f],\ [\pm1,\pm f,\ 0],\ \text{and}\ [\pm f,0,\pm1]$$

A graph of this twelve-cornered solid looks like this-

ICOSAHEDRON WITH V=12, F=20 AND E=30



Since the area of any equilateral triangle of sides s is always $A=s^2\text{sqrt}(3)/4$, the total surface area of the Icosahedron will be $5s^2\text{sqrt}(3)$.

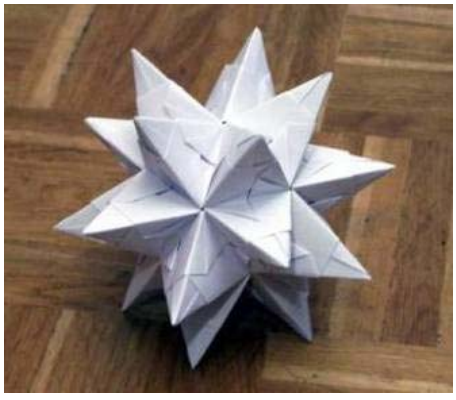
Several years ago I constructed an Icosahedron model from 20 equal sized equilateral triangles made of birch wood and glued together by the use of small wood stabilizer blocks cut to equal the 138.19^0 interior dihedral angle between neighboring surfaces. The edges where filled and smoothed out with wood putty and painted black. The resultant structure looks like this-

WOOD CONSTRUCTION OF AN ICOSAHEDRON

Nov.2009

The structure is quite sturdy and has lasted some five years without any damage despite of considerable handling. Note the perfectly fitting pentagonal column to which the Icosahedron is connected.

Although there are no additional Platonic Solids than the five discussed above, there are many variations involving tessellated polytropes and the replacement of a single polygon faces by several different ones. Good examples of such are soccer balls which have their outer surfaces made of a combination of regular pentagons and regular hexagons. The famous buckyballs of physical chemistry have a near spherical structure and consist of 60 carbon atoms arranged as 20 hexagons plus 12 pentagons. By taking one of the Platonic Solids and placing pyramids above each face or raising a sphere surface in a spatially periodic manner, an infinite number of other 3D solids can be generated. Certain configurations resemble 3D stars and often have a pleasing appearance. Here are two examples-



One of the simplest of such star like polytropes can be constructed by placing pyramids of height h and square base b^2 upon the six square faces of a cube of volume b^3. The volume of such a six-pointed star is-

$$V=b^2\{b+2h\}$$

and its surface area is-

$$S=12bsqrt(h^2+b^2/4)$$

To graph this star via computer one imagines two concentric spheres of radius r_1=bsqrt(3)/2 and radius r_2=h+b/2 and places the eight vertices [±b/2,±b/2,±b/2] on the inner sphere and the six vertices [±(h+b/2,0,0], [0,±(h+b/2),0] and [0,0,±(h+b/2)] on the outer sphere. Connecting neighboring vertices by straight lines then leads to the computer command-

```
with(plots);
polygonplot3d({[[0,0,7],[-1,1,1],[1,1,1],[1,-1,1],[-1,-1,1],[-1,1,1]],[[0,0,-7],[-1,1,-1],[1,1,-1],[1,-1,-1],[-1,-1,-1],[-1,1,-1]],[[0,7,0],[1,1,-1],[1,1,1],[-1,1,1],[-1,1,-1],[1,1,-1]],[[0,-7,0],[1,-1,-1],[1,-1,1],[-1,-1,1],[-1,-1,-1],[1,-1,-1]],[[7,0,0],[1,1,-1],[1,1,1],[1,-1,1],[1,-1,-1],[1,1,-1]],[[-7,0,0],[-1,1,-1],[-1,1,1],[-1,-1,1],[-1,-1,-1],[-1,1,-1]]},axes=none,scaling=constrained,title=` VIEW OF A HEXSTAR`);
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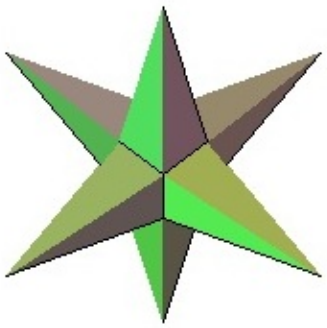
which produces the polytrope shown-

VIEW OF A HEXSTAR



b=2 and h=6

In this figure we have set b=2 and h=6. Setting up the calculation procedure for this and other related tessellated solids is a good exercise in mental 3D visualization involving rotations and symmetry. Notice the shape looks something akin to a medieval mace (flail). These were very effective weapons for close combat because a great deal of momentum could be imparted to such a spiked ball when attached to a swinging rod via a short chain. The momentum carried by its large central mass allowed the spikes to penetrate most chainmail armor. Guns of course made such weapons obsolete.