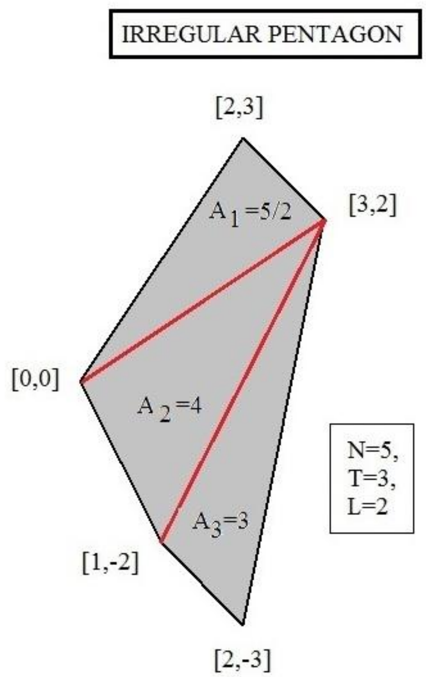
VOLUME OF POLYHEDRA USING A TETRAHEDRON BREAKUP

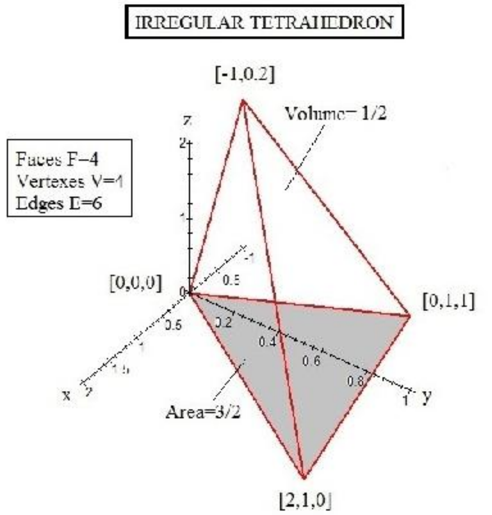
We have shown in an earlier note that any two dimensional polygon of N sides may be broken up into N-2 triangles T by drawing N-3 lines L connecting every second vertex. Thus the irregular pentagon shown has N=5,T=3, and L=2-



With this information, one is at once led to the question-" How can the volume of any polyhedron in 3D be determined using a set of smaller 3D volume elements". These smaller 3D eelements are likely to be tetrahedra . This leads one to the conjecture that –

A polyhedron with more four faces can have its volume represented by the sum of a certain number of sub-tetrahedra.

The volume of any tetrahedron is given by the scalar triple product $|V_1 x V_2 \cdot V_3|/6$, where the three Vs are vector representations of the three edges of the tetrahedron emanating from the same vertex. Here is a picture of one of these tetrahedra-

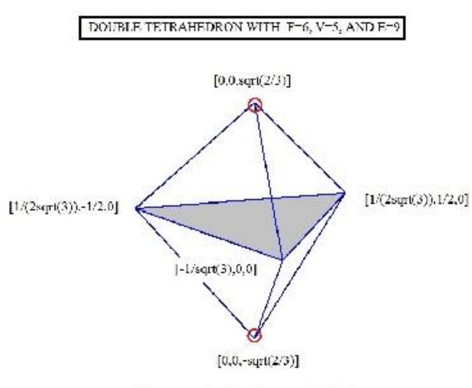


Note that the base area of such a tetrahedron is given by $|V_1 x V_2]/2$. When this area is multiplied by 1/3 of the height related to the third vector one finds the volume of any tetrahedron given by-

$$\text{Vol} = \frac{|(V_1 x V_2)\cdot V_3|}{6} = \frac{Abs}{6}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

, where x,y, and z are the vector components.

The next question which arises is how many tetrahedra are required to completely fill a polyhedron? We can arrive at an answer by looking at several different examples. Starting with one of the simplest examples consider the double-tetrahedron shown-



It is clear that the entire volume can be generated by two equal volume tetrahedra whose vertexes are placed at [0,0,sqrt(2/3)] and [0,0,-sqrt(2/3)]. The volume of each is-

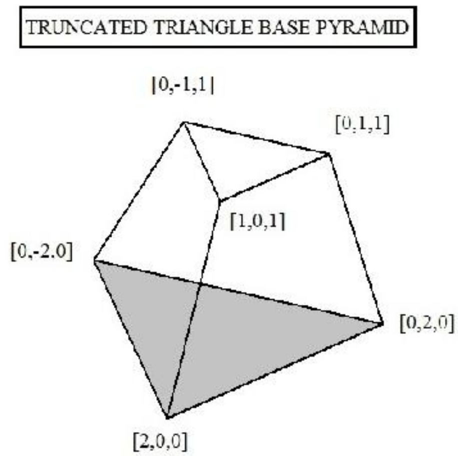
$$\mathrm{Vol}_{tetra}=\frac{Abs}{6}\begin{vmatrix} \frac{3}{2\sqrt{3}} & \frac{-1}{2} & 0 \\ \frac{3}{2\sqrt{3}} & \frac{1}{2} & 0 \\ \frac{1}{2\sqrt{3}} & \frac{-1}{2} & -\sqrt{\frac{2}{3}} \end{vmatrix}=\frac{\sqrt{2}}{12}$$

So the total volume of the double tetrahedron becomes sqrt(2)/6. Notice that it took only two tetrahedra to match all the edges of the double tetrahedron. So it appears that-

The number of tetrahedra required to fill any polyhedron is such that they are able with proper placement cover the entire volume and all edges of the polyhedron without overlap.

Consider next a standard pyramid having a square base and four equilateral side triangle faces. This solid has F=5 faces, V=5 vertexes, and E=8 edges. Since every tetrahedron has E=6 edges, we will need at least two tetrahedra and possibly more. Cutting the pyramid into two equal halves by passing a plane through a base diagonal and the pyramid vertex, we achieve the goal of matching the pyramid volume with just two equal volume tetrahedra. If all edges of the pyramid have unit length, the diagonal equals sqrt(2) and the pyramid height is 1/sqrt(2). This means each tetrahedron has a base area of ½ so that each has volume Vol_{tetra}=(1/2)[1/(3sqrt(2)]. So the pyramid will have a volume twice this amount, namely, 1/[3sqrt(2)]=0.2357.. .

We next look at a more complicated polyhedron in the form of a truncated pyramid with triangular base. A wire-frame depiction of this solid looks as follows-



Vertexes V=6, Faces F=5, Edges E=9

Here the number of edges are E=9. So it will require at least two tetrahedra or possibly more to match its volume and cover each of its 9 edges. Looking a bit more at the figure it is clear that three tetrahedral are needed. We place the first of these using [2,0,0] as its vertex. It has volume-

$$\mathrm{Vol}_1=\frac{Abs}{6}\begin{vmatrix}-1 & 0 & 1\\ -2 & 2 & 0\\ -2 & -2 & 0\end{vmatrix}=\frac{4}{3}$$

Next we look at the second tetrahedron by using [0,2,0] as a vertex. It has volume-

$$\mathrm{Vol}_2=\frac{Abs}{6}\begin{vmatrix}0 & -4 & 0\\ 0 & -1 & 1\\ 1 & -2 & 1\end{vmatrix}=\frac{2}{3}$$

These two terahedra have not yet succeeded in covering all edges of the truncated pyramid. To do so will require a third tetrahedron with vertex at [0,-1,1]. It has volume-

$$\mathrm{Vol}_3=\begin{vmatrix}0 & -1 & -1\\ 0 & 2 & 0\\ 1 & 1 & 0\end{vmatrix}=\frac{1}{3}$$

Now all edges of the former pyramid are covered without any tetrahedron overlap. Hence the truncated pyramid volume is –

$$\mathrm{Vol}_{\mathrm{pyramid}}=4/3+2/3+1/3=7/3$$

That this is the correct answer can easily be verified by subtracting the volume(8/3) of a full pyramid of height 2 and base area 4 from the volume (1/3) from a smaller pyramid of height 1 and base area 1. It should be pointed out that there are other arrangements of tetrahedra which will equal the total volume of the truncated pyramid.

An interesting side-light in connection with truncated pyramids is the fact that the ancient Egyptians already knew over 3000 years ago (see the Moscow Mathematical Papyrus) that the volume of a truncated square base pyramid equals-

$$Vol = \frac{h}{3}(a^2 + ab + b^2)$$

,where h is the truncated pyramid height, a its base side length and b the top square side length. Such a pyramid has F=6, V=8, and E=12. Because of the geometry it will take at least six tetrahedra to match the pyramid volume and cover all the twelve edges. Such an evaluation would be rather lengthy compared to simply treating the truncated pyramid as the difference between two complete pyramids, to produce the truncated pyramid volume-

$$\mathrm{Vol}_{\mathrm{pyramid}} = \frac{a^2 H}{3} - \frac{b^2}{3}(H - h) = \frac{h}{3}\left(\frac{a^3 - ab^2 + b^2}{a - b}\right) = \frac{h}{3}(a^2 + ab + b^2)$$

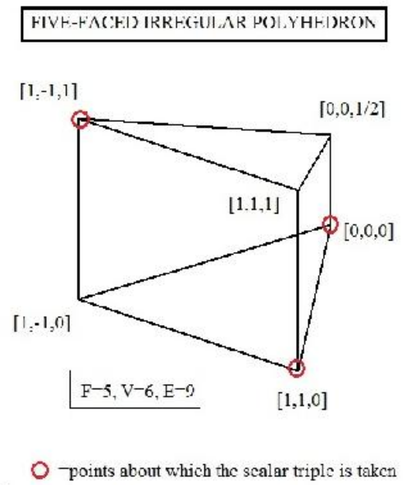
Here H=ah/(a-b) is the height of a complete pyramid of base area a^2. That the ancient Egyptians were able to arrive at this result without the benefit of the binomial expansion and calculus is truly amazing.

We can also use our three vector approach to determine the volume of any pyramid of height H and having a polygon base of N sides each of length s. One finds-

$$Vol_{pyramid} = \frac{N}{6} Abs\begin{bmatrix} 0 & 0 & -H \\ h & -s/2 & -H \\ h & s/2 & -H \end{bmatrix} = \frac{NHs^2}{12}\cot(\pi/N)$$

, where h=(s/2)cot(π/N). Note as $N \to \infty$, we recover the volume for a cone $Vol=\pi R^2 H/3$.

We complete our discussions by looking at the volume of a five face polyhedron depicted by the following wire frame picture-



We have 9 edges and so will probably require two or more tetrahedra to cover all edges and match the polyhedron volume. We try a construction using the three points circled in red as the vertexes of three different tetrahedra. For vertex [1,-1,1] we get a tetrahedron of volume-

$$V_1=\frac{Abs}{6}\begin{vmatrix}0 & 0 & -1\\ 0 & 2 & 0\\ -1 & 1 & -1/2\end{vmatrix}=\frac{1}{3}$$

For the vertex[1,1,0] we get the tetrahedron volume-

$$V_2=\frac{Abs}{6}\begin{vmatrix}0 & 0 & 1\\ -1 & -1 & 0\\ 0 & -2 & 0\end{vmatrix}=\frac{1}{3}$$

And finally, the third tetrahedron with vertex at [0,0,0] has volume-

$$V_3=\frac{Abs}{6}\begin{vmatrix}0 & 0 & 1/2\\ 1 & -1 & 0\\ 1 & 1 & 0\end{vmatrix}=\frac{1}{6}$$

These three tetrahedra are sufficient to cover all 9 edges of the solid and also match the total volume. Adding things together produces the total volume of the five-face polyhedron as 5/6. One could get the same answer by placing the third tetrahedron at [0,0,1/2] for its vertex. The main idea is that the three chosen tetrahedra cover all edges of the original polyhedron. Placing the third tetrahedron at [1,1,1] will not work since the edge between [0,0,0] and [0,0,1/2] will then not be covered by any of the tetrahedra edges. Also note there is no way one could use just two tetrahedra to match this volume.