PRIME, COMPOSITE, AND PERFECT NUMBERS

Several month ago while examining the properties of integers, we came up with a new way to classify various integers $N$ via a quotient defined as the ratio of the sum of all divisors of a number, excluding $N$ and one, divided by the number. That is:

$$f(N) = \frac{[\sigma(N)-(N+1)]}{N}$$

where $\sigma(N)$ is the sigma function of the number $N$. We term this quotient the Number Fraction. As found in books on number theory, one can express the sigma function occurring in this expression as:

$$\sigma(N) = \prod_{k=1}^{n} \frac{p_i^{a_i+1} - 1}{p_i - 1}$$

where $N = p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots \cdot p_n^{a_n}$ represents the prime number breakup of the integer $N$. One need not necessarily require that this product be actually evaluated. The simplest way to determine a number fraction such as $f(12)$, is to simply write $f(12) = \frac{2+3+4+6}{12} = \frac{5}{4} = 1.25$.

Very often one will find that the sigma function is contained within a computer program such as in MAPLE or MATHEMATICA, and so the evaluation of $f(N)$ for smaller numbers $N$ less than about 40 digits is straightforward. For $N=554400$ we find $f(N) = 3.396361833\ldots$. This happens to be the largest $f(N)$ present in the range $1<N<600000$. Also our PC can show that:

$$f(1956478234589421383475091235472873409713) = 0.5238100215\ldots$$

and:

$$f(12345678910111213141516171819202122232425) = 0.2400802015\ldots$$

Several interesting general features of $f(N)$ are observed. First of all, whenever a number $N$ is prime, the Number Fraction must necessarily be zero. This condition was built into our original definition of $f(N)$. Any value of $f(N)$ other than zero indicates a composite number with the degree of compositeness increasing with increasing values of $f(N)$. Semi-primes $N=p_1 \cdot p_2$ have their $f(N)$ near zero and (what I term) large or super composites have $f(N)>1$. The classical definition of a perfect number $N$ is one where the sum of all its divisors except $N$ just equals $N$. That is $\sigma(N)=2N$. In terms of $f(N)$, perfect numbers are given by:

$$f'(N) = 1 - \frac{1}{N}$$

The first few perfect numbers found are $N=6$, 28, 496, and 8128. The $f(N)$s for these numbers lie slightly below unity. In the literature one also finds references to deficient and abundant numbers. In terms of $f(N)$ these have the definitions:
\[
f(N) = \begin{cases} 
> (1 - \frac{1}{N}) = \text{abundant} \\
< (1 - \frac{1}{N}) = \text{deficient}
\end{cases}
\]

So for larger \( N \), a number with \( f(N) > 1 \) is abundant and one where \( f(N) < 1 \) is deficient. \( N=12 \) is an abundant number but \( N=14 \) is deficient. I will refer to those \( N \)s where \( f(N) > 1 \) as large or super-composites. \( N=1260, 9240, 277200, \) and \( 3326242920 \) are all examples of super-composites.

It is possible to work out analytical expressions for certain functional forms of \( N \). Take the case of \( N=2^n \) with \( n=1, 2, 3, \) etc. Substituting into the definition of \( f(N) \) we find-

\[
f(2^n) = 1 - \frac{1}{2^{n-1}}
\]

This leads to \( f(N) \)s of intermediate value near one. For Mersenne Primes, defined as \( N=2^p-1 \) for certain primes \( p \), the value of \( f(N) \) is always zero. Another result is-

\[
f(3^n) = \frac{1}{2} \left( 1 - \frac{1}{3^{n-1}} \right)
\]

which allows us to generalize to-

\[
f(p^n) = \frac{1}{(p-1)} \left( 1 - \frac{1}{p^{n-1}} \right)
\]

for any prime number \( p \). Note that this result will not work when \( p \) is a composite. It also follows that –

\[
f(p^2) = \frac{1}{p} \quad \text{and} \quad f(p^4) = \frac{(p^2 + p + 1)}{p^3}
\]

From this one can further generalize things to obtain the identity-

\[
\sum_{k=1}^{2n-1} \frac{1}{p^k} = f(p^{2n})
\]

We can also use the \( f(p^n) \) result and work backwards to show that the sigma function \( \sigma(p^n) \) equals \( (p^{n+1} - 1)/(p-1) \). This provides an alternate proof for the basic product representation of \( \sigma(N) \) given earlier. In addition we have in the limit as \( n \to 1 \) that \( f(p) = 0 \).
The simplest way to find super-composites is to carry out the following MAPLE computer scheme-

\[
\text{with(numtheory): with(plots): listplot([seq(-c+((sigma(x)-x-1)/x),x=a..b)],view=[a..b, 0..d]);}
\]

where \( a<x<b \) gives the range of \( x \) over which we are plotting the number fractions \( f(x) \), \( c \) is a number greater than 1 for super-primes and \( d \) lies between 0 and 3.

For a graph of the first one hundred Number Fractions we use \( a=1, b=100, c=0 \) and \( d=2 \). It produces the listplot-

Notice in this range of \( x \) the largest composite is located at \( x=60 \) and has a number fraction of \( f(60)=1.783333... \) To determine the largest values of \( f(x) \) in a given range, we typically choose a \( c \) greater than about 1. This avoids all the hash appearing for smaller \( f(x) \) yet clearly points out the location of super-composites. We have carried out such calculations and find the following results for the largest \( f(x) \)s in the ranges indicated-

<table>
<thead>
<tr>
<th>Range</th>
<th>Value of ( x ) at max ( f(x) )</th>
<th>Number Fraction, ( f(x) )</th>
</tr>
</thead>
</table>
| 1<x<100 | 60                              | 1.78333333

![Graph of Number Fractions](image)
It is seen that the value of the largest super-composite seems to increase very slowly with increasing $x$ but spot checks of $\lim_{n \to \infty} f(N^n)$ indicate that $f(x)$ is probably bounded. Certain regularities are observed including –

$$\lim_{n \to \infty} \left\{ \frac{f(2^n)}{n} \right\} = 1, \quad \text{and} \quad \lim_{n \to \infty} \left\{ \frac{f(6k^n)}{n} \right\} = 2$$

which hold for most positive integers $k$. A spot check with the twelve digit number $N=113210697600$ yields the Number Fraction $f(N)=4.335823268$. Here $N-1=(1088533)\ (104003)$ is a semi-prime while $N+1=113210697601$ is a prime number. Note that for all the $x$s considered above, the value of $x$ at a local max $f(x)$ is divisible by 6.

We can view the detailed structure of $x$ versus $f(x)$ by plotting a narrow range about one of these maxima. Take the case of $x=55440$ and its ±10 neighborhood-The curve looks like this-

SUPER-COMPOSITE AT $X=55440$ WHERE $F(X)=3.1869949$. (Note the twin primes at 55439 and 55441)
Note that an approximate mirror reflection occurs about the super-prime at $x=55440$ and also observe the appearance of a double prime at $x=55440 \pm 1$. The spike in $f(x)$ is very reminiscent of a rogue wave occurring in a turbulent sea. The observation that primes (or possibly semi-primes) are likely to occur in the immediate neighborhood of a large or super-prime has already been noted by us in an earlier article including the observation that all primes above $p=3$ can be represented as $Q$ primes defined as $Q=6n \pm 1$. It seems that it is statistically likely that the number in the immediate neighborhood of a large or super-composite is likely to be a prime or semi-prime number, although this will become less likely as $N$ becomes very large since the spacing between primes increases with increasing $N$.

As an example, take the case of the super-composite $x=665280=3080 \cdot 6^3$ which has $f(x)=3.398266895$. It suggests that either $x+1$ or $x-1$ or both may be prime or semi-prime. Testing things out we find-

$$\text{ifactor}(665281)=577 \cdot 1153 \quad \text{and} \quad \text{ifactor}(665279)=665279$$

so that 665279 is indeed a prime number while 665281 is a semi-prime composed of the primes 577 and 1153. The value of $f(665281)=0.0026004049..$ and so is not quite zero. We also observe that large local peaks in $f(x)$ are generated by $x=3080 \cdot 6^k$, with $k$ a positive integer. This suggests there will be many primes of the form $x=3080 \cdot 6^k \pm 1$. Among those primes are-

$$x:=3080 \cdot 6^{11} - 1 = 1117414932479$$

$$x:=3080 \cdot 6^{43} - 1 = 8893126766801959400651827451296481279$$

$$x:=3080 \cdot 6^{117} + 1 = 3406018569457934597628147355783263837479000839037894646007197472348848463746299874908181626881$$

$$x:=3080 \cdot 6^{152} + 1 = 5855187066761506298864797939364842668487632579032451525970270599107957711517856662244784711592670849738572021498983093370881$$

We can also generate very large primes by finding that number $n$ where-

$$N=6(\text{Random number}+n) \pm 1$$

is prime. Random numbers can be generated by multiplying various irrational numbers together or just generating them directly with a random number generator. As an example, we looked at the number-

$$N=60((2578398250129356753468308953331+n)-1)$$

and found it becomes prime when $n=31$. That is-

$$N=1547038950077614052080998537201719$$
is a 33 digit long prime number. We verify this fact by noting \( f(N) = \frac{\sigma(N) - N - 1}{N} = 0 \) as expected.

April 1, 2013