EVALUATION OF THE INTEGRAL

\[
\int_{x=0}^{1} \frac{P_{2n}(x)}{1 + x^2} \, dx = N_n + M_n \frac{\pi}{4}
\]

In a previous note above we examined the properties of Legendre polynomials and in the process found that integrals of the form-

\[
\int_{x=0}^{+1} \frac{P_{2n}(x)}{a^2 + x^2} \, dx = N_n + M_n \frac{1}{a} \arctan\left(\frac{1}{a}\right)
\]

For the special case of \(a=1\), one obtains the following identity for \(\pi\)-

\[
\pi = 4 \frac{M_n}{n} \left[ -N_n + \frac{1}{0} \frac{P_{2n}(x)}{1 + x^2} \right]
\]

Running through the first few integers \(n\) one finds-

\[
\pi = 4 \frac{\left(-\frac{3}{2}\right)}{2} \left[ - \frac{1}{0} \frac{P_2(x)}{1 + x^2} \right] = 3 + \left[ 3 - \pi \right] \quad \text{for} \quad n = 1
\]

\[
\pi = 4 \frac{\left(\frac{20}{3}\right)}{(17/2)} \left[ + \frac{1}{0} \frac{P_4(x)}{1 + x^2} \right] = \frac{160}{51} + \left[ - \frac{160}{51} + \pi \right] \quad \text{for} \quad n = 2
\]

and-

\[
\pi = 4 \frac{\left(-\frac{161}{5}\right)}{41} \left[ + \frac{1}{0} \frac{P_6(x)}{1 + x^2} \right] = \frac{644}{205} + \left[ - \frac{644}{205} + \pi \right] \quad \text{for} \quad n = 3
\]

What is clear from these results is that the first term on the right of the equality gets ever closer to the true value of \(\pi\) while the second term gets progressively smaller approaching zero as \(n\) goes to infinity. This means that one has a way to approximate \(\pi\) by simply looking at-

\[
\pi \approx \lim_{n \to \infty} \left[ -\frac{4N_n}{M_n} \right]
\]
The constants $M_n$ in this evaluation are simply the remainder term in long division of $P_{2n}(x)$ by $x^2+1$ while the $N_n$s are gotten from an integration from $x=0$ to 1 of the polynomial arising from division of $P_{2n}(x)$ by $x^2+1$ without the remainder. After these constants have been found one simply takes the ratio $-4N_n/M_n$ to obtain the approximation for $\pi$. The whole procedure is straight forward to carry out via your PC. One generates the $P_n(x)$ polynomials using the generating formula-

$$(n + 2)P_{n+2} = (2n + 3)xP_{n+1}(x) - (n + 1)P_n$$ \hspace{1cm} \text{with} \ P_0(x) = 1 \text{ and } P_1(x) = x$$

In MAPLE the entire program reads-

(1)- \( P[0]:=0; P[1]:=1; \)

(2)- \text{for n from 0 to 140 do} \ P[n+2]:=-((n+1)*P[n]+x*(2*n+3)*P[n+1])/(n+2)od:

(3)- \text{n:=?; } M[n]:=\text{rem}(P[2*n],x^2+1,x);

(4)- \text{}N[n]:=\text{int}(\text{quo}(P[2*n],(1+x^2),x),x=0..1);

(5)- \text{PIAPPROX= evalf(-4*N[n]/M[n],100);}

The question mark is to be replaced by the value of $n$ being used. The calculations lead to the following results-

<table>
<thead>
<tr>
<th>$n$</th>
<th>PIAPPROX</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3…</td>
</tr>
<tr>
<td>5</td>
<td>3.141592…</td>
</tr>
<tr>
<td>10</td>
<td>3.14159265358979…</td>
</tr>
<tr>
<td>25</td>
<td>3.1415926535897932384626433832795028841…</td>
</tr>
<tr>
<td>50</td>
<td>3.1415926535897932384626433832795028841971693993751058209749445923078164062860…</td>
</tr>
<tr>
<td>66</td>
<td>3.141592653589793238462643383279502884197169399375105820974944592307816406286208998628034825342117068…</td>
</tr>
</tbody>
</table>

We have cut off the approximations at the decimal point where they first depart from $\pi$. Note the accuracy increases by about one to two decimal points per unit increase in $n$. At $n=25$ one first exceeds the Ludolph result of 35 correct decimal places found by him back in the sixteen hundreds using the Archimedes method and which took him a lifetime of calculations. One first reaches a 100 digit accuracy at $n=66$. The quotient producing this result is-
\[
\pi \approx -4N_{66} M_{66} = -4 \int_{x=0}^{1} \operatorname{quo}(P_{132}(x), x^2 + 1, x) dx = \frac{A B}{C D}
\]

\[
\pi = 3.1415926535897932384626433832795028841971693993751058209749\ldots
\]

where-

\[
A = 16344307100151766275719569624919542395219172089931202926028057567103\ldots
\]

\[
B = 73786976294838206464
\]

\[
C = 280886967533990987019681903704641652707036307939791678353350629238707
\]

\[
D = 13666734749380163413296024576419304586504627516675222125
\]

Another approach to approximating \( \pi \) is to expand the \( \arctan(1/a) \) terms in multiple \( \arctan \) formulas for \( \pi \). We begin with our own formula discussed earlier-

\[
\pi = 48 \arctan(\frac{1}{38}) + 80 \arctan(\frac{1}{57}) + 28 \arctan(\frac{1}{239}) + 96 \arctan(\frac{1}{268})
\]

and write-

\[
\arctan(\frac{1}{a}) = \frac{a}{M[n,a]} \left[-N[n,a] + \int_{x=0}^{1} \frac{P_{2n}(x)}{(a^2 + x^2)} dx \right]
\]

Substituting into the above four term arctan formula, one has the \( \pi \) approximation-

\[
\pi \approx -1824 \frac{N[n,38]}{M[n,38]} - 5460 \frac{N[n,57]}{M[n,57]} - 6692 \frac{N[n,239]}{M[n,239]} - 25728 \frac{N[n,268]}{M[n,268]}
\]

This time, since the ‘a’s are large numbers, the convergence will be quite rapid. Taking \( n=27 \), we find the same 100 digit accurate result as given above but this time requiring a considerably smaller value of \( n \) to achieve this.
A still faster convergence rate may be accomplished using arctan formulas for \( \pi \) containing only larger values of ‘a’. A way to create such formulas is to make use of the identity:

\[
\arctan\left(\frac{1}{a}\right) = 2 \arctan\left(\frac{1}{2a}\right) - \arctan\left(\frac{1}{4a^3 + 3a}\right)
\]

Then starting with the Machin Formula:

\[
\pi = 16 \arctan\left(\frac{1}{5}\right) - 4 \arctan\left(\frac{1}{239}\right)
\]

and applying the above doubling arctan formula several times leads to the nine term arctan formula:

\[
\pi = 1024 \arctan\left(\frac{1}{320}\right) - 8 \arctan\left(\frac{1}{478}\right) - 16 \arctan\left(\frac{1}{515}\right) - 32 \arctan\left(\frac{1}{4030}\right)
\]

\[
- 64 \arctan\left(\frac{1}{32060}\right) - 128 \arctan\left(\frac{1}{256120}\right) - 256 \arctan\left(\frac{1}{2043240}\right)
\]

\[
- 512 \arctan\left(\frac{1}{16384480}\right) + 4 \arctan\left(\frac{1}{54608393}\right)
\]

This time our approximation for \( \pi \) is:

\[
\pi \approx -1024 \frac{N[n,320]}{M[n,320]} - 8 \frac{N[n,478]}{M[n,478]} - 16 \frac{N[n,515]}{M[n,515]} - 32 \frac{N[n,4030]}{M[n,4030]} - 64 \frac{N[n,32060]}{M[n,32060]}
\]

\[
- 128 \frac{N[n,256120]}{N[n,256120]} - 256 \frac{N[n,2048240]}{M[n,2048240]} - 512 \frac{N[n,16384480]}{M[n,16384480]} + 4 \frac{N[n,54608393]}{M[n,54608393]}
\]

At \( n=8 \) it already produces a value for \( \pi \) correct to 100 decimal places. The very rapid convergence clearly stems from the fact that the smallest value of ‘a’ used is the large number \( a=320 \).

It appears that one could use this approach to find \( \pi \) to accuracies of a million and even a billion decimal places. Unlike AGM methods it would require no root taking and involves only very simple integrations of \( 2(n-1) \) order polynomials \( P_{2n}(x)/(x^2+a^n) \) over the range \( 0<x<1 \).

October 2009