POWERS OF REAL AND COMPLEX NUMBERS

As most of you learned earlier in your introductory algebra class, the power \( p \) of a complex number –

\[
N = z = x + iy = \sqrt{x^2 + y^2} \exp i \arctan(\frac{y}{x}) = r \exp i \theta
\]

is given by-

\[
z^p = r^p [\cos(p \theta) + i \sin(p \theta)]
\]

In terms of geometry this power represents a complex number located at radius \( R = r^p \) and at angle \( p \theta \). Thus \( z = i^{57} = \exp[2\pi(28)+\pi/2] = i \) is located in the complex \( z \) plane at one unit from the origin along the \( y \) axis. The number \( z \) will be real and positive if \( \theta = 2\pi n \) and is real and negative if \( \theta = (2n+1)\pi \). It is pure imaginary if \( \theta \) is \( \pm (2n+1)\pi/2 \).

The usual rules of multiplication and division of powers of numbers continues to hold for complex numbers. Thus-

\[
(1 + i)^{23} \cdot (1 - i)^{11} = (1 + i)^{12} \cdot 2^{11} = 2^{17} i^6 = -131072
\]

and-

\[
i^{5(1+i)} = i^5 \cdot \exp(i\pi 5i / 2) = i \exp(-5\pi / 2) = i(0.000388203..)
\]

In these evaluations we made use of the following facts-

\[
i^n = \exp(in \pi / 2)
\]

\[
z^{p_1} \cdot z^{p_2} = z^{(p_1+p_2)}
\]

\[
\text{abs}(z) = \sqrt{[(x + iy)(x - iy)]}
\]

and also gave an example where the power \( p = \sigma + i\tau \) is complex.

Note that the power \( p \) need not necessarily be integer. Take for example the function-

\[
z^n = (1 + i)^n \text{ with } 0 < n < \infty
\]
This yields $z_1=1+i$, $z_2=2i$, $z_3=-2(1+i)$, etc. The absolute value of $z$ which equals the locus radius in the $z$ plane thus increases in a monotone manner $\sqrt{2}$, 2, $2\sqrt{2}$, etc suggesting that $z_n$ traces out a spiral pattern in the $z$ plane. We have-

$$r = \text{abs}(z_n) = 2^{(n/2)} = \exp\left(\frac{n}{2} \ln 2\right) \quad \text{and} \quad \theta = \frac{n\pi}{4}$$

Eliminating $(n/2)$ we find-

$$r = \exp\left(\frac{2\ln 2}{\pi} \theta\right)$$

This is just the logarithmic spiral of Bernoulli. At $\theta=2\pi$ it yields $r=16$ as expected. Here is its graph-

Bernoulli was so fascinated by this spiral that he had it engraved on his tombstone. You can still see it by visiting the large church overlooking the Rhine river in Basel, Switzerland. I remember visiting the place several years ago and noticing that the main attraction was the grave of Erasmus of Rotterdam who is also buried there.

One notices that when a complex number is plotted as a point in the $z$ plane, its polar representation is-

$$z = r \exp(i\theta) = r \exp[i(\theta + 2\pi k)] \quad k = \pm 1, \pm 2, \pm 3, \text{etc}$$
This means the angle portion of the number is $2\pi$ periodic which in modular arithmetic language says that the answer in a power calculation will remain unchanged when $\theta$ is replaced by $\theta \pmod{2\pi}$. We have, for instance, that

$$i^{279} = i \cdot \exp i (70 \cdot 2\pi - \pi) = -i$$

This result is even easier to establish by noting that –

$$i^p = i^{p \pmod{4}}$$

so that $i^{279} = i^{80} \cdot i^{-1} = -i$

Next let us find the value of $p$ for which the number-

$$N = [(a + ib)(b + ia)]^p$$

is real and positive.

when $a$ and $b$ are real positive integers. We have considered this type of number in an earlier note (June 2009) and note that it simplifies to the form-

$$N = (a^2 + b^2)^p \cdot i^{p \pmod{4}}$$

Thus any even power $p$ will make $N$ real. Thus if $a=8$, $b=6$ and $p=50$, we find that $N = -10^{160} = -1$ googol.

We next concentrate on the powers of real numbers such as $N = B^p$ with $p$ being a real integer and $B$ the real base. The brute force method of evaluating this number is to perform $p-1$ multiplications of $A$. That however is not the most efficient way to accomplish things. The optimum approach using the fewest number of multiplications is accomplished as follows. We demonstrate the procedure by looking at-

$$N = 2^{217}$$

Rewrite this number as follows-

$$N = 2^{100+100+20-3} = 4^{110} / 8 = 2 \cdot 256^{27}$$

Thus we have reduced things from 216 multiplications to just 27 to yield the answer-

$$N = 210624583337114373395836055367340864637790190801098222508621955072$$
This type of calculation can be expedited by constructing a binary power tree based upon the power p being used. Taking the case of $N=A^{27}$ we first construct the following tree:

The tree shows essentially one way to break up the number 27 by having each node having two offspring whose sum matches that of the parent. At the third generation the tree produces the identity:

$$A^{27} = A^5 \cdot A^5 \cdot A^5 \cdot A^5 \cdot A^2 = A^2 \cdot (A^5)^5$$

Thus –

$$2^{27} = 4 \cdot 32^5 = 134217728 \quad \text{and} \quad 3^{27} = 9 \cdot 243^5 = 7625597484987$$

The answer is here produced by six multiplications instead of the 26 required in a brute force evaluation.

Finally let us work out the value of the number:

$$N = -\frac{i^{38}(1+i)^{64}}{2^{32}}$$
Employing the various operations discussed above we have-

\[ N = \frac{1 \cdot 2^{32} \cdot \exp(16i\pi)}{2^{32}} = 1 \]