RICH NUMBERS

If one looks at the first N integers 1,2,3,4,5,.. N it is noted that some of these have a large number of divisors, and especially so, when the number is a multiple of $6=2\cdot 3$ or $60=2^2\cdot 3\cdot 5$. Thus we have that-

N= 24 has eight divisors {1, 2, 3, 4, 6, 8, 12, 24}

, while its neighbor N=25 has only three divisors {1, 5, 25}

and N=23 has just two divisors {1,23}. We can call a number with a large number of divisors compared to its neighbors a **rich number.** Thus 24 is a rich number. Any number with just two divisors will be a **prime number** such as N=31 which has {1,31} as its divisors. The number N=60(127)=7620 is also a rich number with its neighbor N=7621 being a prime.

There are several ways to determine the number of divisors a number has. The first and most elementary way is to simply run through a set of integers 1 through approximately the square root of N and note down the number n which make N/n an integer and then write down the number pairs [n , N/n] for which this is true. So, for example, we have N=16 produces [1,16],[2,8],[4,4] so we have the divisors {1,2,4,8,16}. Thus N=16 has five distinct divisors. Mathematically we can express this result as $\tau(16)=5$, where $\tau(N)$ is the tau function of number theory. Many computer programs have the tau function already built into them, so that $\tau(N)$ can be written down instantaneously. Thus $\tau(24)=8$, $\tau(360)=24$, $\tau(7620)=24$, and $\tau(34739)=2$. Note that the last of these corresponds to a prime number p since $\tau(p)$ is always equal to 2 for primes.

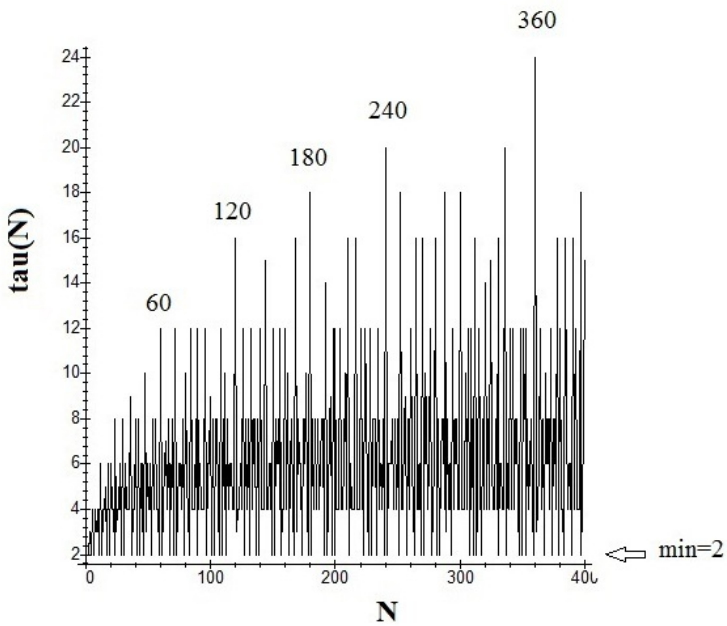
Another way to determine the value of $\tau(N)$ is to note first of all that $\tau(p^k)$ equals 1+k when p is a prime and k a specified positive integer. Since any number can be written as-

$$N=\prod_{m=1}^{M} p_m{}^{k_m}$$

We have that-

$$\tau(N)=\prod_{m=1}^{M}(1+k_m)$$

Thus we find, in agreement with the above, that $\tau(360)=(1+3)(1+2)(1+1)=24$ since $360=2^3\cdot 3^2\cdot 5$. We also find the neighboring numbers yield $\tau(359)=2$ and $\tau(361)=3$. This clearly indicates that N=360 is a rich number. A plot of N versus $\tau(N)$ in the range 2 through 400 yields the following picture-

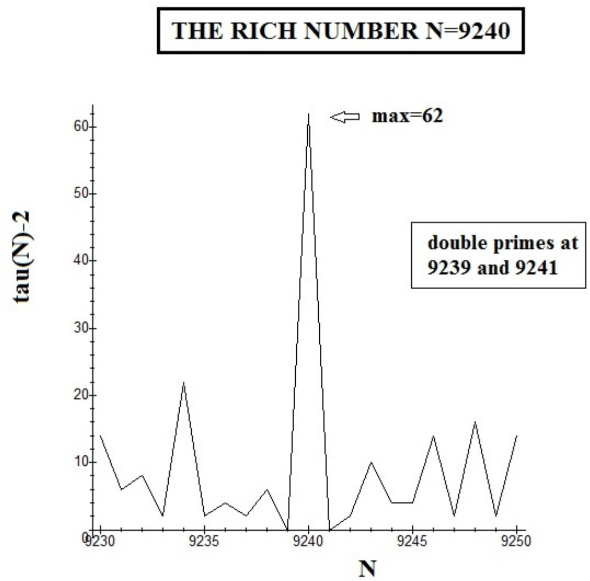


The graph shows that certain numbers have much larger tau values than their immediate neighbors and hence can be considered rich. Clearly 60,120,180, 240, and 360 are such numbers in the range considered. $\tau(N)$ is a relatively slow increasing function of N so that the largest value of tau found in $2<N<10{,}000$ is just $\tau(9240)= 64$.

To get rid of the value $\tau(p)=2$, we can introduce the slightly modified function-

$$g(N)=\tau(p)-2=[(k_1+1)(k_2+1)(k_3+1)\ldots] -2$$

Plotting g(N) versus N in the neighborhood of N=9240=60(154) produces the result-



Note that τ(9240)-2 has a value of 62 while its neighbors N=9239 and N=9241 are primes with τ(N)-2 equal to zero. Here N=9240 is clearly a rich number . It has the prime number product form-

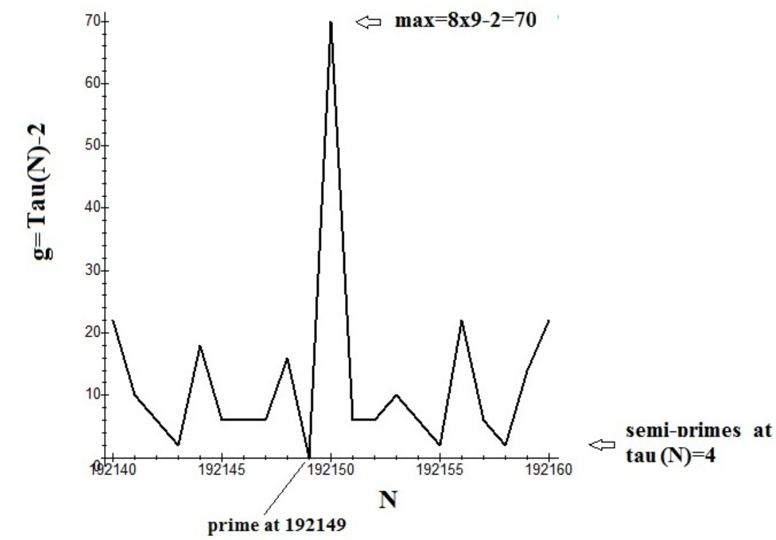
$$N = 9240 = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11$$

Which produces $\tau(9240)=(3+1)(1+1)^4=64$, so that τ(N)-2=62 as shown in agreement with the above graph. If we go back and look at $N=24=2^3\cdot 3$ we have τ(24)=(3+1)(1+1)=8 which is also a rich number in its neighborhood. Using the forms of these last two rich numbers, we can postulate that rich numbers will have the property that their prime number expansions have the form-

$$N = 2^{k_1} \cdot 3^{k_2} \cdot 5^{k_3} \cdot .... \textit{ with the the number of primes kept small}$$

Let us test this conjecture out for the number $N=192150=2\cdot 3^2\cdot 5^2\cdot 7\cdot 61$ where N has seventy-two distinct divisors since τ(N)=(1+1)(2+1)(2+1)(1=1)(1+1)=72. A plot of N versus τ(N)-2 in the range 192140<N<192160 follows-

$$N=192150=2\times3^2\times5^2\times7\times61$$



Note the spike in tau at the rich number location and the much smaller values of tau in the immediate neighborhood of N. Since τ-2=0 at 192149, the number must be a prime. Values where τ=4 are semi-primes such as 192155=5 x 38431

One notices that often (but not always) the number in the immediate neighborhood of a rich number will be a prime. We will call such primes S Primes as these form a subset of all possible primes. To demonstrate, consider the rich number $N=2^{17}\cdot3^{11}\cdot5=116095057920$. It has a very sharp spike with value $\tau(N)-2=430$. Here we find a S Prime at –

$$N+1=116095057921$$

Another example where one has an S Prime occurs for $N=2^{31}\cdot3^{13}-1=$ 3423782572130303. Here 3423782572130304 is a rich number. Note that $N=2^p$,where p is any prime, is another example of a rich number. It happens to be the one leading to the Mersenne Numbers and Primes $M=2^p-1$.

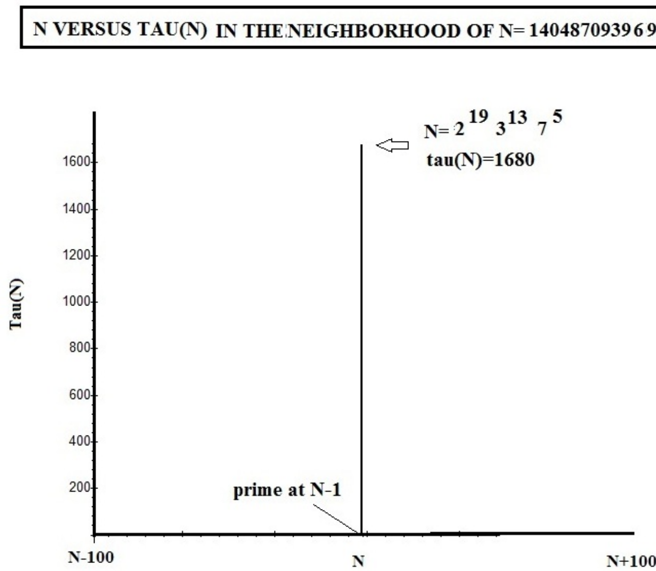
Some other S Primes are-

$$2^{11}\cdot3^5\cdot5-1= 2488319$$

$$2^{13}\cdot3^7\cdot7^5\cdot11^3-1= 400780868640767$$

$$2^{19}\cdot3^{13}\cdot7^5-1= 14048709396922367$$

To show you how large $\tau(N)$ can become relative to its neighbors look at the following graph of N versus Tau(N) for the large rich number $N=2^{19}\cdot 3^{13}\cdot 7^{5}$ =14048709396922368



The neighbors of N have such low value that they don’t even register on the graph.

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