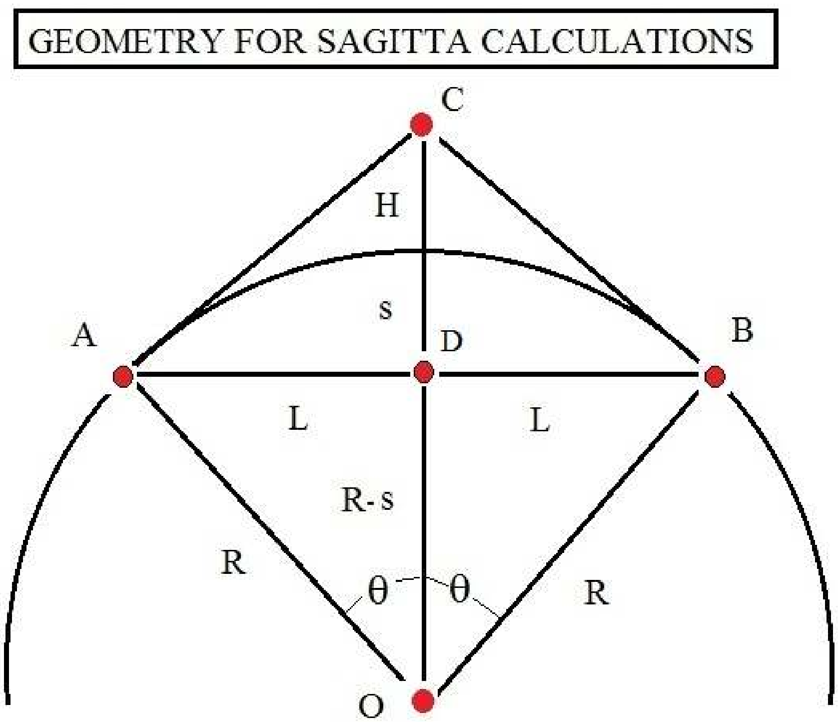
THE SAGITTA AND ITS APPLICATIONS

One of the more important geometric quantities associated with the line of sight distance on a curved surface, with the maximum depth of a straight line tunnel bored through the earth between two points, and with the depth of circular arcs in architecture is the Sagitta. It is defined as the distance 's' shown on the following diagram-



Chord length=2L, Circle radius=R, Sagitta=s, Height above sealevel=H, Apothem=R-s, Arclength=2θR

The diagram shows a circle (or sphere) of radius R plus its arclength S=2Rθ between two surface points A and B. The chord length is given by 2L=2Rsin(θ). Making use of the right triangles OCB, ODB, and DCB, we have from the Pythagorean Theorem that-

$$CB = \sqrt{H(2R+H)} \quad , \quad L^2 = s(2R-s) \quad , \quad and \quad (s+H)^2 = CB^2 - L^2$$

Introducing the non-dimensional quantities σ=s/R, λ=L/R, α=H/R and β=CB/R, we can convert these non-linear algebraic equations into the following forms-

$$\beta^2 = \alpha(2+\alpha) \quad , \quad \lambda^2 = \sigma(2-\sigma) \quad and \quad \beta^2 = \lambda^2 + (\sigma+\alpha)^2$$

In adition, we have tan(θ)=(α+σ)/λ from the above figure. If we now ask what is the relation between the line of sight BC=βR and the height H=αR above sea level on the earth's surface, we find from the quadratic formula that-

$$\alpha = -1 + \sqrt{1+\beta^2} \approx \frac{\beta^2}{2} - \frac{\beta^4}{8} + O(\beta^6)$$

Retaining only the first term in the expansion, one obtains the well known Sagitta Formula that the line of sight distance CB viewed from a ship at height H above the water surface is-

$$CB \approx \sqrt{2HR}$$

Sailors are quite familiar with this formula and it is the reason the crows nest on old time sailing ships where placed at the highest possible point on the ship. At a height of H=100ft a seaman can peer out a distance of just sqrt(200x3960x5280)=64666.52 ft=12.247miles when the water surface is smooth. The distance one can see will actually be larger if one is looking at something which rises above the water level such as the cliffs of Dover in England when viewed from Calais in France. I remember several years ago looking toward England from the D-Day landing beaches in France and could see no land across the channel. This was clearly due to the one hundred miles separating the two countries there. The effectiveness of water skimming missiles such as the Exocet is that line of sight radars are unlikely to detect such missiles in time to initiate evasive maneuvers.

Let us next look at another problem which can be treated by the above results. Consider digging a straight tunnel from point A to point B on the earth. Its length will be the chord 2L and the maximum depth reached below the earth's surface will be the Saggitta value of s. In non-dimensional terms we have-

$$\sigma = 1 - \sqrt{1-\lambda^2} \approx \frac{\lambda^2}{2} + \frac{\lambda^4}{8} + O(\lambda^6)$$

Thus the first approximation shows that the tunnel length relates to the maximum depth as-

$$2L = AB \approx 2\sqrt{2sR}$$

If one were interested in building such a tunnel connecting say New York City with Washington DC we would be dealing with a distance as the crow flies of 2L= 204 miles and a mean earth radius of R= 3960 miles. The tunnel would pass about s=1.3 miles below Philadelphia. If one could eliminate all friction for an object passing through such a tunnel, it would require no energy expenditure. The object would start from rest at one of the cities, accelerate till reaching a maximum speed at the half way point and then decelerate until coming to rest at the other city. That is, the object undergoes Simple

Harmonic Motion. The trip would take about 42.5 minutes. We will consider more detailed properties of such tunnels in a future note including safety, acceleration, heating problems, and, most importantly, cost.

A third problem we can discuss using the above results is what portion of the earth will be visible by someone viewing from a distance H above its surface. Clearly one expects to see a circular region whose radius is related to the angle θ where this angle can be thought of as the polar angle in a spherical coordinate system. The total surface area of the earth is $4\pi R^2$ and the spherical cap whose edge contains the points A and B has surface area $2\pi R^2$([1-cos(θ)]. The ratio of viewed area compared to half the total sphere area works out to be-

$$Ratio = [1-\cos(\theta)] = \sigma = \frac{\alpha}{1+\alpha}$$

A spy satellite moving at an elevation of H=500 miles above the earth's surface has an α=500/3960=0.126. so that it can see about 11 percent of the half surface $2\pi R^2$ of the earth at any one time. We observe essentially the entire front face of the full moon when viewed from earth since there α is about 60.

Finally I leave you with the question "How far out into the Pacific Ocean can one see on a clear day from the top of Mt. Mauna Loa in Hawaii? This is an easy calculation involving the value of CB= βR for α=13,680/(3960x5280)=0.00065427. The problem becomes a bit more complicated if one asks how many miles away one can detect an incoming jet flying at 32,000 ft via radar based on the top of Mauna Loa at 13,680ft elevation?