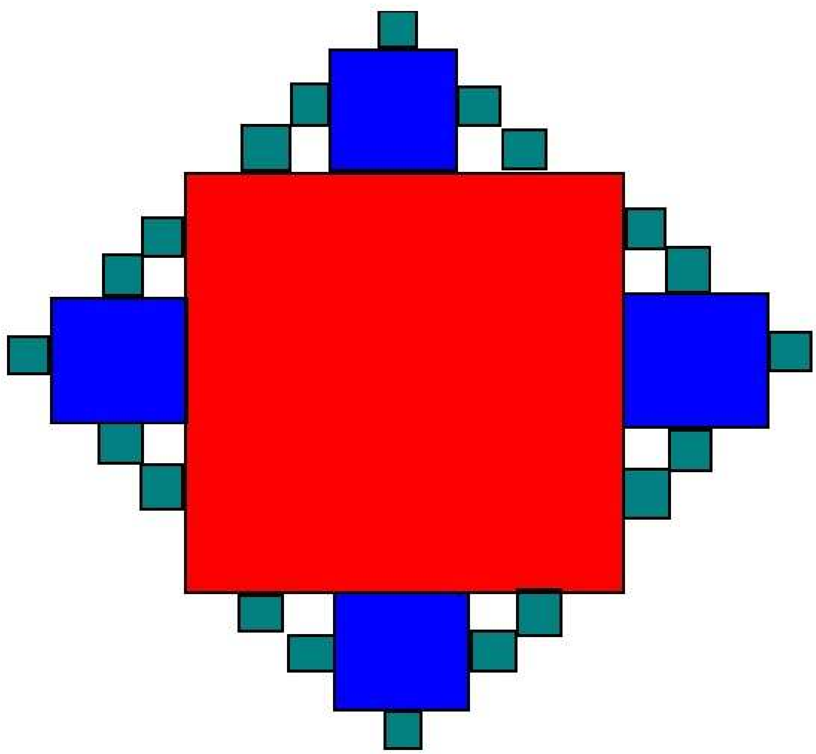
GENERATION OF SELF SIMILAR 2D PATTERNS

A recent NOVA television program(Aug 24,2011) presented an interesting discussion on fractals. It was shown how fractals and their property of self-similarity can be used to describe several different phenomena occurring in nature. Of all the fractal concepts perhaps one of the easiest to understand is the Koch curve in which a straight line is continually divided following a certain rule. We examine here a bit more of the self-similar properties encountered with Koch curves and the Mandelbrot set. Specifically we look at constructing self-similar 2D patterns based on the same concept used in the construction of the well known Koch Snowflake.

The starting point of our discussion will be a square of unit sides. We first apply a simple iteration rule of placing four smaller but identically shaped squares of side length 1/3 along the middle of each of the four sides of the original square. This constitutes the first iteration. Next we apply a second iteration in which squares of side-length 1/9 are placed in the middle of the 20 line segments found after the first iteration. The result will be a pattern as shown in the accompanying figure-

DEMONSTRATING THE SELF-SIMILARITY OF A KOCH SQUARE



Area(red)=1 Area(blue)=4/9 Area (green)=20/81

I have used different colors to better indicate the squares formed after each iteration. This process can be continued indefinitely with the squares having ever smaller side-length of $(1/3)^n$ and increasing number $N=4(5)^{n-1}$ .Here n represents the order of the iteration. Thus after three iterations each square will have an area $A_3=1/81$ and there will be 100 of these squares. If the iteration is continued indefinitely the total area covered by all the self-similar squares will be-

$$A_{total} = 1 + 4[\frac{1}{9} + \frac{5}{9^2} + \frac{25}{9^3} + ...] = 1 + \frac{4}{5}\sum_{k=1}^{\infty}\left[\frac{5}{9}\right]^n = 2$$

That is, eventually the entire region within a square of side-length sqrt(2) will be filled with self-similar squares of ever smaller side-length. This fact is already obvious when examining just the original square and the first and second iterations.

When looking at this problem from a biological view point, one can say that each iteration represents a new generation of squares which in turn beget their own next generation. What is important is that all the squares are self similar in the sense that all generations of squares retain the same basic shape. At the tenth generation there will be a total of 7812500 new identical squares of area 1/3486784401 each descended from original unit side-length square.

A modification of the above approach for generating subsequent generations is to attach a new generation n of squares only to the non-covered sides of the previous n-1 generation squares. This time the total area becomes-

$$A_{total} = 1 + \frac{4}{9}\left[1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + ...\right] = \frac{5}{3}$$

As expected, this area is just a little less than that found for the previous case. It also departs from the standard Koch generating approach. It is, however, a more realistic model when considering the problem as one of generation growth.

It is not always necessary that the iteration involves smaller self-similar areas. The generation of the same size squares on an original existing square followed by the attachment of two squares for each square of the previous generation will end up covering the entire x-y plane without gaps. The same will be true for hexagons with two new hexagons attached to each previous generation hexagon as shown-

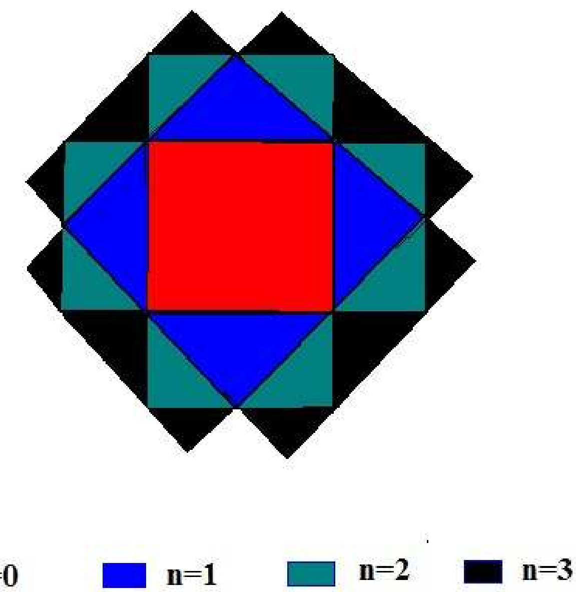
Here the first generation produces six identical hexagons of identical size. The second generation consists of 12 hexagons identical to the first. Thus the nth generation will consist of $3(2)^n$ hexagons. Each new generation will have twice the number of hexagons as the previous. If the original red hexagon has area $A_0=1$, then the total area covered by all generations up through the nth will be-

$$A_{total} = 1 + 6(1 + 2 + 4 + ... + 2^{n-1}) = -5 + 6 \cdot 2^n$$

Thus eventually the hexagons will cover an infinite plane without any overlaps. This hexagon pattern is a property of snow flakes indicating that their origin is due to its hexagonal shaped water molecule initiator. Here is one of the famous photos by amateur scientist W.A.Bentley(1865-1931) of a snowflake-

Another construction covering the entire x-y plane without gaps involves self- similar triangles generated from an original square as shown in the following figure-

GENERATIONS 1-3 OF SELF-SIMILAR RIGHT TRIANGLES FORMED ON A SQUARE



Here the original unit side-length square(or two triangles) has four 45-90-45 degree triangles placed upon its four sides as shown. The hypotenuse of these triangles (colored blue) equals 1. These four triangles represent the first generation n=1. Next the second generation n=2 will consist of eight similar right triangles of smaller hypotenuse length 1/sqrt(2). These smaller triangles are shown in green . Continuing on, we find that the third generation n=3 consists of 16 self-similar right triangles of hypotenuse length [1/sqrt(2)]2=1/2 placed upon the green triangles . A quick generalization indicates that the nth generation will have-

$$triangle\ number = 2^{n+1} \quad hypotenuse\ length = \left[\frac{1}{\sqrt{2}}\right]^{n-1}$$

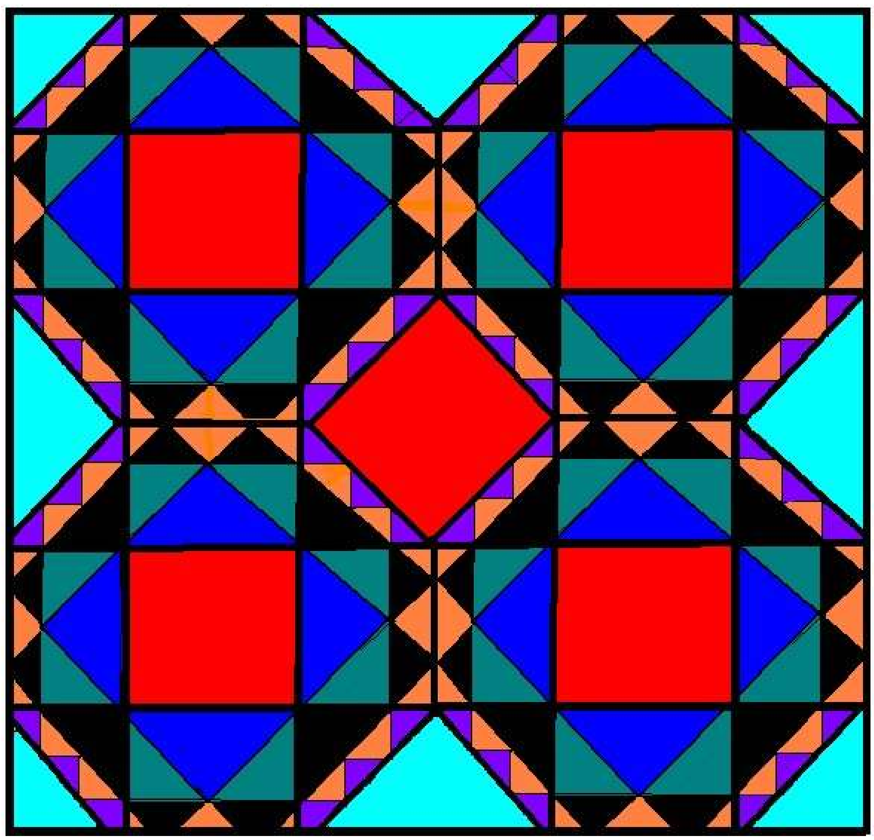
The total area covered by this pattern up through the nth generation is-

$$A_{total} = 1 + 4(\frac{1}{4}) + 8(\frac{1}{8}) + 16(\frac{1}{16}) + ... = 1 + n$$

There is no limit to the size of area which could be covered using these triangles as triangular tiles.

Repetitive tile patterns can also be created by using the above triangle pattern as a periodic combination. Below I show you such a combination I have constructed-

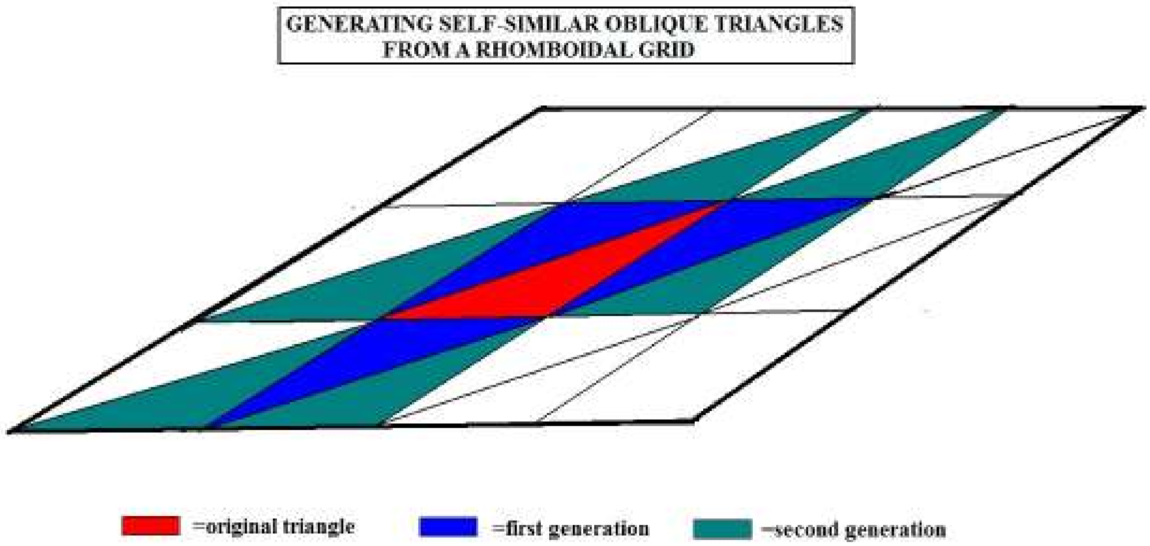
TILE OR RUG PATTERN BASED ON SEF-SIMILAR TRIANGLES



It would make an interesting wall assembly or pattern for a Persian rug.

Finally we address the question of weather or not it is possible to construct a self-similar pattern based an oblique triangle where the sides a, b, and c are of different length. We already know from the above examples that equilateral triangles(six of these make up a hexagon) and isosceles right triangles(two of these make a square) can produce non-gap patterns. What about a pattern using an oblique triangle where non of its three sides or angles are equal?

In thinking a bit about this possibility, we came up with an answer in the affirmative. Our starting point is to recognize that any two identical oblique triangles can be used to form a Rhomboid as shown in the following figure-



Extending this Rhomboid to a rhomboidal grid allows one to generate three identical first generation triangles shown in blue. These are followed by six identical green triangles of the second generation. From Hero's Formula, we know that each triangle has an area-

$$A_{triangle} = \sqrt{s(s-a)(s-b)(s-c)} \quad where \quad s = \frac{1}{2}(a+b+c)$$

Each new generation produces exactly 3n new identical sized images of the original triangle shown in red. If one where interested in tiling the x-y plane with such triangles, one can be assured that there will be no gaps present. I believe I may have run across a simpler right triangle patterns in the wood flooring of one of the European palaces. I can't quite remember if it was Versailles, Chenonceau , Schoenbrunn or Nympenburg.

August 27, 011