## RELATION BETWEEN NUMBER SEQUENCES AND DIFFERENCE EQUATIOINS

The first exposure to elementary mathematics occurs for children at the pre--kindergarten level when they first learn to count numbers one-two-three etc. That is , they master the sequence-

$$S(n)=\{1,2,3,4,5,6,7,8,9,10, \ldots n \ldots\}$$

This form can be expressed abstractly as the difference equation-

$$f[n+1]=f[n]+1 \quad \text{subject to } f[1]=1$$

Its solution is f[n]=n. There exist an infinite number of other sequences which may be expressed as difference equations. Some of these will form the topic of the present article.

We begin with an extended form of the above sequence by looking at-

$$S(n)=\{1,2^p,3^p,4^p,5^p,\ldots,n^{p,}\ldots.\}$$

, where p is a positive integer power. The difference equation here reads-

$$f[n+1]=f[n]+(n+1)^p-n^p \quad \text{subject to } f[1]=1$$

For p=2 it yields -

$$f[n+1]=f[n]+2n+1 \quad \text{subject to } f[1]=1$$

The solution for this last difference equation is $f[n]=n^2$and so represents the sequence involving the square of all the integers. It tells us at once that $26^2=625+51=676$ and $101^2=1000+201=10201$.

Another difference equation occurs for p=3. It reads-

$$f[n+1]=f[n]+3n^2+3n+1 \quad \text{subject to } f[1]=1$$

Solving, produces the sequence-

$$S(n)=\{1,8,27,64,125,\ldots,n^3,\ldots\}$$

and the solution $f[n]=n^3$. From it we see that $11^3=1000+300+30=1=1331$.

For the next sequence let us look at-

$$S(n)=\{1,2,3,5,8,13,21,34,\ldots\}$$

It is equivalent to the difference equation-

$$f[n+2]=f[n]+f[n+1] \quad \text{subject to } f[1]=1 \text{ and } f[2]=2$$

It represents the famous Fibonacci Sequence first introduced by the Italian mathematician Leonardo Fibonacci (1170-1250). This sequence has been extensively studied since that time. It was Johannes Kepler of astronomy fame who first showed that the ratio of f[n+1]/f[n] approaches the value of {1+sqrt(5)}/2 as n goes to infinity. That is, the ratio equals the golden ratio $\varphi=1.61803398\ldots$ The French mathematician Edouard Lucas (1842-1891) modified the Fibonacci Sequence slightly by changing the starting conditions to f[1]=1 and f[2]=3. This produces the sequence-

$$S(n)=[\ 1,3,4,7,11,18,29,...\ \}$$

The numbers in this sequence are known as the Lucas Numbers. The ratio of f[n+1]/f[n] does not change from the golden ratio as n goes to infinity.

One can come up with additional variations on the Fibonacci Sequence. For example, we could look at the sequence-

$$S(n)=\{1,2,3,6,11,20,37,\ldots\}$$

Without looking below, can you recognize what the next term in the sequence will be? A little reflection will show that here we have the difference equation-

$$f[n+3]=f[n+2]+f[n+1]+f[n] \quad \text{subject to } f[1]=1,\ f[2]=2 \text{ and } f[3]=3$$

So the next term in the sequence is 68 followed by 125 and then 230. The ratio of f[n+1]/f[n] here has the unique value of –

$$R=1.839286755214161132551852564653286600424\ldots$$

as n gets large.

One can generate an infinite number of other sequences by starting with a difference equation subjected to specified initial conditions. Take the example of-

$$f[n+1]=f[n]+(n+1) \text{ subject to } f[1]=1$$

This solves as f[n]=n(n+1)/2 and produces the sequence-

$$S(n)=\{1,3,6,10,15,21,28,..\}$$

The nth term in his sequence just equals the sum of the first n integers. That is, f(7)=28=1+2+3+4+5+6+7.
Making the following small variation on this last equation we get-

f[n+1]=f[n]+$(n+1)^2$ subject to f[1[=1

This produces the sequence-

S(n)= {1,5,14,30,55,…}

Now we notice that 5=1+4, 14=1+4+9, and 30=1+4+9+16. So the elements in the sequence just represent the sun of the squares of n up to n. The actual solution can be found by solving the matrix equation-

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ 27 & 9 & 3 & 1 \\ 64 & 16 & 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 14 \\ 30 \end{bmatrix}$$

It produces the values A= 1/3, B=1/2, C=1/6 and D=0. Hence we get the solution-

f[n]=(n/6){(n+1)(2n+1)}

It tells us that the sum of the squares of the first 100 integers is 338350.
By looking at the terms (n+1) and $(n+1)^2$ involved in the summation of the first and second powers of n, we can generalize and say the equation-

f[n+1]=f[n]+$(n+1)^p$ subject to f[1]=1

has as its sequence elements the sum of the pth power of the integers up through n. The solution is-

$$f[n] = 1 + 2^p + 3^p + .... + (n-1)^p + n^p = \sum_{k=1}^{n} k^p$$

There are an infinite number of other difference equations capable of generating number sequences. Sometimes finding the next integer in a partially completed sequence can become quite challenging especially if creation of the sequence involves some non-standard mathematical manipulations. One of these is the following sequence posed on a recent NPR radio program and conveyed to me by Dr. Haftka , a colleague of mine at the University of Florida. The first seven terms of the proposed sequence reads-

S(n)=1,2,4,8,16,23,28,….

On inspection the first five elements follow a number doubling, but this fails with 23 and 28. Also there is no new information obtained by looking at the first 1,2,4,8,7,5 and second 1,2,4,-1,-2 differences. One can, however, find the subsequent terms in this sequence by letting –

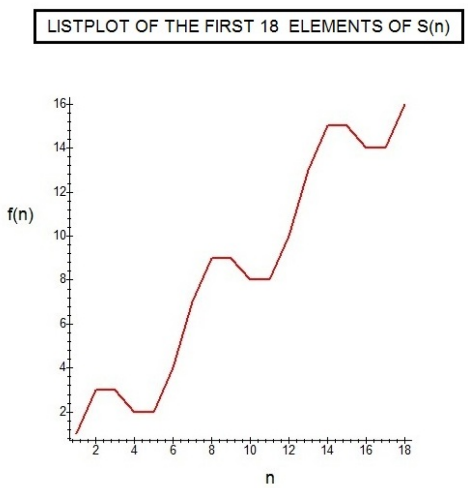
f(n+1)=f(n)+number of ten units in f(n) +number of one units in f(n)

Thus we have f(5)=8+0+8=16, f(6)=16+1+6=23, f(7)=23+2+3=28 and the next is f(8)=28+2+8=38. Continuing on we get 38+3+8=49 and 49+4+9=62 etc. I have had no luck so far in finding an equivalent difference equation for this sequence and am wondering if it even exists.

Finally, I leave the reader with the following sequence-

S(n)={1,3,3,2,2,4,7,9,9,8,8,10,13,15,15,14,14,16,_, _ …} .

Can you find the next two integers in this sequence and also the difference equation corresponding to S(n)? The elements relate to each other via standard math operations. Here is a listplot of the first 18 elements f(n) -




The first difference should yield a clue to the values of elements f[19] and f[20].

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