GENERATING NUMBER SEQUENCES

A standard sequence can be represented by a horizontal line of elements f[n] , where f[n] is expressed as a function of n. One of the simplest of such sequences is given by the formula-

$$f[n+1] = f[n]+1 \quad subject \quad to \quad f[1]=1$$

, where the square bracket represents the subscript of f. The formula produces the positive integer sequence-

S={1, 2, 3, 4, 5, 6, 7, 8, 9,…}

In other words, all positive integers. Another very simple sequence is given by the function f[n]=sin(nπ/2) which produces-

S={1, 0, -1, 0, 1, 0, -1, 0, 1, 0, …}

Here the elements are cyclic with the four element sequence {1 ,0, -1, 0} repeating forever. Historically , perhaps the best known sequence is the Fibonacci sequence introduced by Leonardo of Pisa in his 1202 book "Liber Abaci ". This sequence reads-

S={1, 2, 3, 5, 8,13, 21, 34,…}

and is defined by the formula-

f[n+2]=f[n+1]+f[n] with f[1]=1 and f[2]=2

The above examples represent just three of an infinite number of other sequences. We want here to examine some these other, not so obvious , sequences.

Our starting point will be those sequences defined by the iteration formula-

f[n+1]=F(f[n]) subject to f(1)=1

, where F represents a specified function of f[n]. Take the sequence-

S={ 1, 2, 5, 26, 677,…}

What is the next number? It is clear that things grow very rapidly and that $2=1^2+1$ $5=2^2+1$, $26=5^2+1$, and $677=26^2+1$. This fact at once produces the generating formula-

$$f[n+1]=f[n]^2+1 \qquad with \quad f[1]=1$$

the next term in the sequence will be f[6]=458330. The function F(f[n]) is here $f[n]^2+1$.

Another very important sequence is-

S={1, 2, 6, 24, 120, 720}

Most readers will recognize that here f[n]=n!, the factorial of the integers 1 , 2, 3, 4, 5,…etc.. That is, f[1]=1, f[2]=2, f[3]=6, etc.. We have the generating formula -

$$f[n+1\}=(n+1)!=\prod_{k=1}^{n+1} k$$

Since the division of two factorials produces a positive integer as long as the term in the numeratore is larger than the denominator, one can produce numerous other sequences based on n factorials. One interesting one which we have come up with defines a new sequence by-

$$f[n]=\sum_{k=0}^{n-1}\frac{(n+k)!}{(n-k)!}=[\frac{n!}{n!}+\frac{(n+1)!}{(n-1)!}+...+\frac{(2n-1)!}{1!}]$$

This yields-

S={1, 7, 133, 5421, 383911, 41793403, 6473328681, …,. }

Note that the elements in this sequence are bounded from below by (2n-1)!

Another sequence with very rapidly increasing elements is-

f[1]=1, f[2]=2 , f[3]=5, f[4]=3126

Can you find the next term f[5] ? Hint, it is a huge number of over eleven thousand digits.

To generate the sequence of the powers of any integer N we can use the generating formula-

f[n+1]=Nf[n] with f[1]=1

The cubes of the first six integers will read-

{1, 3, 9, 27, 81, 243}

It is not always necessary to have the f[n]s be all real or have only positive signs. For example , the relation-

$f[n+1]=i^n$ with f[1]=1 produces the cyclic sequence-

S= {1, i, -1, -i, 1, i, -1, -i, 1,…}

The nth term in S will equal f[n]=cos(πn/2)+isin(πn/2) showing that all elements lie on a unit radius circle in the complex plane.

Another complex sequence follows from the generating formula-

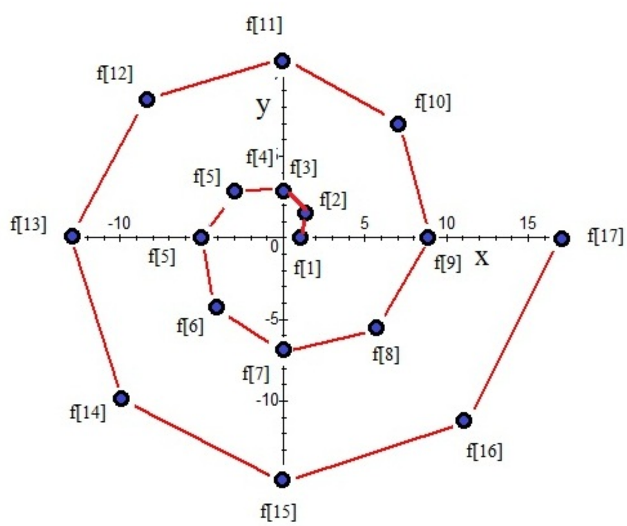
$$f[n]=n\ \exp\{(i\pi(n-1)/4\}$$

This produces the sequence-

$$S=\ \{1,\ (1+i)\sqrt{2},\ 3i,\ (-1+i)2\sqrt{2},\ ....\ \}$$

Here the magnitude of each element is just n. A plot of this sequence in the z=x+iy plane produces the spiral-

ELEMENTS OF THE SEQUENCE f(n)=n exp{i π (n-1)/4} IN THE Z PLANE



Several tears ago we introduced the concept of a number fraction defined by-

$$f[n]=\frac{\sigma(n)-n-1}{n}$$

, where σ(n) is the sigma function of number theory representing the sum of all integer divisors of n. What makes the number fraction of interest is that will vanish only if n is a prime p. Thus f[2]=f[3]=f[5]=f[7]=f[11]=0 but f[4]=1/2 , f[6]=5/6, f[8]=3/4, f[9]=1/3, f[10]=7/10, and f[12]=5/4. A number fraction of zero thus yields the sequence-

$$\{\ 2,\ \ 3,\ \ 5,\ \ 7,\ \ 11,\ \ \ldots\}$$

, which are just the prime numbers. We can pick up the primes in this sequence over any number range, say from 990 through 1010 , by carrying out the one line computer program-

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f:=(sigma(n)-n-1)/n;  for n from 990 to 1010 do {n, f}od;
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It yields just three primes 991, 997, and 1009 in this interval. Note that a mod(6) operation on each of these primes equals 1. This is to be expected considering that all primes above p=3 have the form $6n\pm1$.

To conclude our discussion, we look at a sequence whose elements stay confined between 1 and 3 for all n. This sequence is generated by-

$$f[n+3]=f[n+2]-f[n+1]+f[n] \quad \text{subject to } f[1]=1,\ f[2]=2, \text{ and } f[3]=3$$

It reads-

$$S=\{1,\ 2,\ 3,\ 2,\ 1,\ 2,\ 3,\ 2,\ 1,\ 2,\ 3,\ 2,\ 1,\ 2,\ 3,\ \ldots\}$$

The sequence is cyclic with 1,2,3,2 being the repetitive sub-group. One can state that the element 4k-3 always equals 1, the element 4k-1 always produces 3, and 2 is produced by 4k-2 or 4k. So the 99^{th} element in S will be 3.