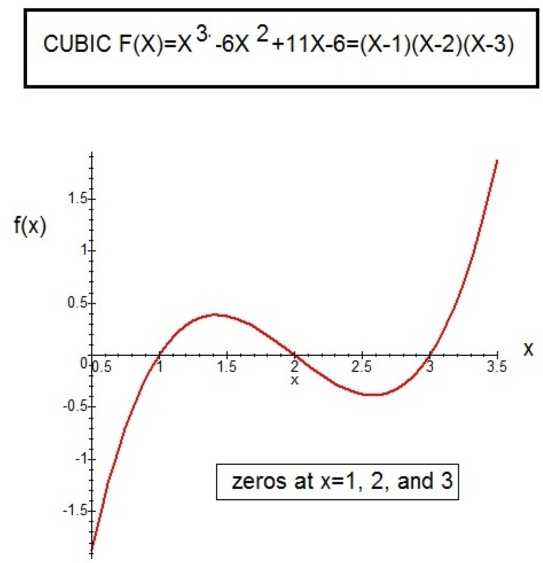
RELATING INFINITE SERIES TO INFINITE PRODUCTS

INTRODUCTION:

It is well known that any polynomial f(x) can be represented by products involving its roots. Thus the cubic-

$$f(x) = x^3 - 6x^2 + 11x - 6 = (x-1)(x-2)(x-3)$$

, with its roots at x=1, 2 and 3. Plotting this polynomial over the range -0.5<x<3.5 produces the following picture-



From it on can make the generalization that –

Any polynomial f(x) may be represented by a product form involving [1-x/x_n], where x_n represents roots of the polynomial.

This observation should continue to hold even if the polynomial has an infinite number of roots such as the sine and cosine functions. It suggests the possibility of re-writing some infinite series into infinite products as first clearly recognized by Leonard Euler several centuries ago. It is our purpose here to re-derive some of the better known relations between infinite series and infinite products and also add a few more identities.

CONVERTING INFINITE SERIES TO INFINITE PRODUCTS:

Consider the function f(x)=sin(x)/x where f(0)=1. This function has simple zeros at x=nπ for integer n in the range -∞<n<infinity. So we can write-

$$\sum_{k=0}^{\infty}\frac{(-1)^n x^{2k}}{(2k+1)!} == \prod_{n=1}^{\infty}[1-(\frac{x}{n\pi})^2]$$

Writing out a few terms on each side of this equality yields-

$$1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+...=[1-(\frac{x}{\pi})^2][1-(\frac{x}{2\pi})^2][1-(\frac{x}{3\pi})^2][...$$

If we now collect all terms multiplying x^2 on both sides of the equality one finds-

$$-\frac{1}{3!}=-\sum_{n=1}^{\infty}(\frac{1}{\pi n})^2$$

This result produces the famouss Euler formula-

$$\frac{\pi^2}{6}=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

This slowly convergent infinite sum is also known as the zeta function of two, namely, ς(2)=1.644934… .

We can also derive some other formulas from the infinite series and infinite product forms of f(x)= sin(x/x. For instance setting x=π/2 leads to the identity-

$$\frac{2}{\pi}=(1-\frac{1}{4})(1-\frac{1}{16})(1-\frac{1}{36})(...=\prod_{k=1}^{\infty}[1-\frac{1}{(2k)^2}]=0.6366197...$$

which is essentially equivalent to the famous Wallis Formula of 1650-

$$\frac{\pi}{2}=\frac{(2\cdot 2)(4\cdot 4)(6\cdot 6)(8\cdot 8)...}{(3\cdot 5)(5\cdot 7)(7\cdot 9)(9\cdot 11)...}$$

Also setting x=π/6 produces-

$$\frac{3}{\pi}=\prod_{k=1}^{\infty}[1-\frac{1}{(6k)^2}]\text{=0.9549296}$$

Taking the product of the last two equations produces the interesting new result-

$$\frac{6}{\pi^2}=\prod_{k=1}^{\infty}[1-\frac{5}{18k^2}+\frac{1}{144k^4}]=0.607927...$$

Other trigonometric functions with an infinite number of simple zeros can also have their infinite series form expressed as an infinite product.

The next most obvious one of these is f(x)=cos(x). Here we get-

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}=\prod_{k=1}^{\infty}[1-(\frac{2x}{(2n-1)\pi})^2]$$

Looking at just the coefficients of x^2 on both sides of this equality, produces-

$$\frac{\pi^2}{8}=1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+...=\sum_{k=0}^{\infty}\frac{1}{(2n+1)^2}$$

Take as a third example f(x)=cos(x)^2. This function has zeros at π(2n+1)/2 and a value of one at x=0. We can express this function as-

$$\cos(x)^2=1-x^2(\frac{\sin(x)}{x})^2=1-x^2\prod_{k=1}^{\infty}[1-(\frac{x}{n\pi})^2]^2$$

With a little manipulation, this result at x=π/3 produces the result-

$$\pi=\frac{3}{\sqrt{\frac{4}{3}\prod_{k=1}^{\infty}(1-\frac{2}{9k^2}+\frac{1}{81k^4})}}=3.14159265...$$

We can also consider x=π/6 where cos(π /6)=sqrt(3)/2. This results in a much improved Wallis formula –

$$\frac{\pi}{3}=\frac{(6^2)(12^2)(18^2)(...}{(6^2-1)(12^2-1)(18^2-1)(...}=\prod_{k=1}^{\infty}[\frac{(6n)^2}{(6n)^2-1}]$$

A final function which we consider is-

$$f(x)=\frac{\tan(x)}{x}=1+\frac{1}{3}x^2+\frac{2}{15}x^4+\frac{17}{315}x^6+O(8)$$

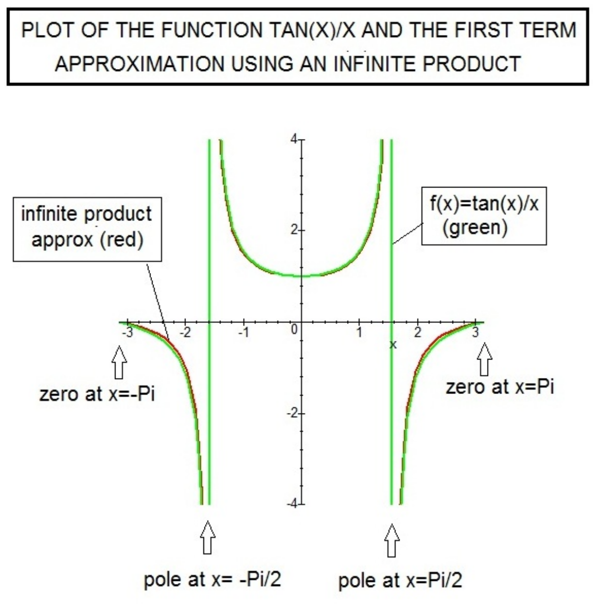
This function has zeros at x=±nπ and poles at x=(2n-1)π/2. We can easily express f(x) as the quotient of two infinite products as follows=

$$\frac{\tan(x)}{x}=\frac{[\frac{\sin(x)}{x}]}{\cos(x)}=\frac{\prod_{k=1}^{\infty}[1-(\frac{x}{n\pi})^2]}{\prod_{k=1}^{\infty}\{1-[\frac{2x}{(2n-1)\pi}]^2\}}$$

One can obtain a rough approximation for tan(x)/x by looking at just the first term of the infinite product quotient. It reads-

$$\frac{\tan(x)}{x}\approx\frac{[1-(\frac{x}{\pi})^2]}{[1-(\frac{2x}{\pi})^2]}$$

Plotting the function f(x) and its approximation near x=0 yields the result shown in the following graph-



The approximation is seen to already yield relatively good agreement with arctan(x)/x for x in the range -3,x,3.

CONCLUDING REMARKS:

One can express many functions f(x) both as infinite series or infinite product. This dual nature also continues to hold for those functions expressible by finite length series and finite products. In general one has-

$$f(x)=\sum_{n=0}^{b} c_n x^n=\prod_{k=1}^{a}[1-(\frac{x}{x_k})]$$

, where a and b can be finite or infinite, f(0)=1, and x_n is a root of f(x). Thus, for instance,-

$$4f(x)=x^3-x^2-4x+4=(x+2)(x-1)(x-2)$$

, where the roots of the function are located at x= -2, 1, and 2. If a function has no real roots then a product representation will not be possible.

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