EVALUATION OF FUNCTIONS BY ITERATION

Have you ever wondered how your pocket calculator or canned mathematics programs on your PC obtains values for functions such as exp-2.5 and cos(3)? The answer clearly must have to do with some form of iteration being carried out on the basic series representation of the function and then evaluated at a given point to any desired order of accuracy. Let us demonstrate such a procedure for the simple exponential function \( F(x) = \exp(x) \) at \( x = 1 \). One starts with the basic infinite series definition-

\[
\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{(2n + 1 + x)x^{2n}}{(2n + 1)!}
\]

\[
= \sum_{n=0}^{\infty} \left[ \frac{(4n + 3)(4n + 2)(4n + 1 + x) + (4n + 3 + x)x^2}{(4n + 3)!} \right] x^{4n}
\]

The second and third sums represent the function defined by terms in the sum taken two and four terms at a time, respectively. Looking at the two term sum, we can define a partial sum-

\[
S[N] = \sum_{n=0}^{N} \frac{(2n + 1 + x)x^{2n}}{(2n + 1)!} \quad with \quad S[0] = (1 + x)
\]

One can then zero in on the value of \( \exp(x) \) via the iteration-

\[
S[k + 1] = S[k] + \frac{(2k + 3 + x)x^{2k+1}}{(2k + 3)!}
\]

Carrying out the first three iterations by hand, we find at \( x = 1 \) that-

\[
S[0] = 2, \quad S[1] = \frac{8}{3}, \quad S[2] = \frac{163}{60}, \quad and \quad S[3] = \frac{685}{252}
\]

Using my computer, the 20\(^{th}\) iteration \( S[20] \) yields the 47 digit accurate result for \( \exp(1) \) of-

\[
S[20] = 2.7182818284590452353602874713526624977572470937
\]
Consider next the arctangent function –

\[
\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(4n+3) - (4n+1)x^2}{(4n+1)(4n+3)} x^{4n+1}
\]

This is a notoriously slowly convergent series when \( x \) is greater than unity because it lacks a factorial term in its denominator. To enhance the convergence by iteration for larger \( x \) it pays to first apply the \( x \) reduction formula:

\[
\arctan(x) = 2\arctan\left(\frac{-1 + \sqrt{x^2 + 1}}{x}\right)
\]

Thus, if wanting to find \( \arctan(1) \), one can look at \( 2\arctan(\sqrt{2}-1) \) and use the iteration:

\[
S[k+1] = S[k] + \left\{ \frac{(4k+7) - (4k+5)(\sqrt{2} - 1)^2}{(4k+5)(4k+7)} \right\} (\sqrt{2} - 1)^{4k+5}
\]

with:

\[
S[0] = \frac{2}{3} (2 - \sqrt{2})
\]

If one now looks at the 30\(^{th}\) iteration \( S[30] \) and multiplies things by 8, the result reads:

\[
8S[50] = 3.14159265358979323846264338327950288419716939937
\]

which is a 48 digit accurate value for \( 4\arctan(1) = \pi \).

As a third example of an iterative solution, consider the sine integral \( \text{Si}(x) \) defined as:

\[
\text{Si}(x) = \int_{t=0}^{x} \frac{\sin t}{t} dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}
\]

Here the iteration procedure reads-
\[ S[k + 1] = S[k] - \frac{(-1)^k x^{2k+3}}{(2k + 3)(2k + 3)!} \quad \text{with} \quad S[0] = x \]

The first few iterations become:

\[
\begin{align*}
S[0] &= x \\
S[1] &= S[0] - \frac{x^3}{3 \cdot 3!} \\
S[2] &= S[1] + \frac{x^5}{5 \cdot 5!} \\
S[3] &= S[2] - \frac{x^7}{7 \cdot 7!}
\end{align*}
\]

Automating the procedure on my computer, we come up with the function \( F(x) = S[50] \). Plotting this we find-
Finally, let us look at finding the approximate value of one of the most elementary irrational numbers \( \sqrt{2} \). We can start with the identity:

\[
(\sqrt{2} - 1)(\sqrt{2} + 1) = 1
\]

which suggests the iteration:

\[
S[n + 1] = 1 + \frac{1}{1 + S[n]} \quad \text{with} \quad S[0] = 1
\]

This is a rather slowly converging iteration yielding only 7 digit accuracy at \( S[10] \). To improve this convergence rate take the square and a fourth power of the above \( \sqrt{2} \) formula. This produces the equalities:

\[
\sqrt{2} = \frac{3}{2} - \frac{1}{6 + 4\sqrt{2}} \quad \text{and} \quad \sqrt{2} = \frac{17}{12} - \frac{1}{12(17 + 12\sqrt{2})}
\]

Using the second equality we have the iteration:

\[
S[n + 1] = \frac{17}{12} - \frac{1}{12(17 + 12S[n])] \quad \text{with} \quad S[0] = 0
\]

At \( n=9 \), this yields the 31 digit accurate value:

\[
S[10] := 1.414213562373095048801688724209
\]

for the \( \sqrt{2} \). An even more rapid convergent iteration follows by taking the eighth power of the original \( \sqrt{2} \) identity shown above. It produces the iteration:

\[
S[n + 1] = \frac{577}{408} + \frac{1}{408(577 + 408S[n])} \quad \text{with} \quad S[0] = 0
\]

and will produce a hundred digit accurate value for \( \sqrt{2} \) at \( S[17] \).

I leave it for the reader to try the iteration-
\[ S[n + 1] = \frac{A}{B} - \frac{1}{B(A + BS[n])} \quad \text{with} \quad S[0] = 0 \]

where A=88673108897 and B=627013566048. You will find this is a superfast converging iteration toward the \sqrt{2}.

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