SHORTEST PATH CONNECTING A SOURCE AND N SINKS

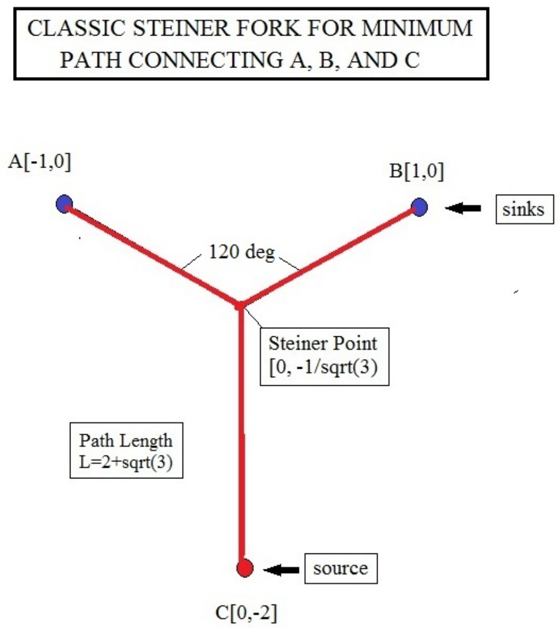
IIt is well known that the shortest distance between a point A[x_A,y_A]) and a point B[x_B,y_B] is a straight line whose length is-

$$L = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}$$

When dealing with three or more points the problem becomes more complicated. As first shown by Jacob Steiner(1796-1863), the minimum path connecting three points (say one source and two sinks) is in the form of a two prong distorted fork where the points A and B lie at the end of the prongs and are connect to the third point C at the bottom of the fork. The point where the three straight line paths meet is known as the Steiner Point. Let us demonstrate the form of minimu path connecting A, B, and C for the special case A[-1.0], B[1,0] and C[[0,-2] . Here we have the path length-

$$L = \sqrt{(-1-0)^2 + (0-y)^2} + \sqrt{(1-0)^2 + (0-y)^2} + \sqrt{(0-0)^2 + (2+y)^2}$$
$$= 2\sqrt{1+y^2} + (2+y)$$

This has a minimum at y=-1/sqrt(3) to yield an optimum path length of L=2+sqrt(3)=3.732.. . A picture of the path follows-



You can see why we call this a Steiner Fork. Originally I wanted to call it Steiner's Martini but concluded fork is a lot more descriptive although less sexy. The main characteristic of the Steiner Fork is that the three radial lines converge at an intersection known as the Steiner Point with the angles between the radial lines always remaining at 120 deg =2π/3 rad. This continues to hold as long as the three points don't lie along the same straight line. One of the more interesting Steiner Forks is where the Steiner Point is at the center of an equilateral triangle with points A, B, and C located at the vertexes. Under this condition the triangle centroid coincides with the Steiner Point. This is usually not the case for other point locations. .
Note that one can think of the above path as an optimum irrigation configuration connecting a water supply (source C) with two spraying points (sinks at A and B). We will make use of this source-sink designation below.

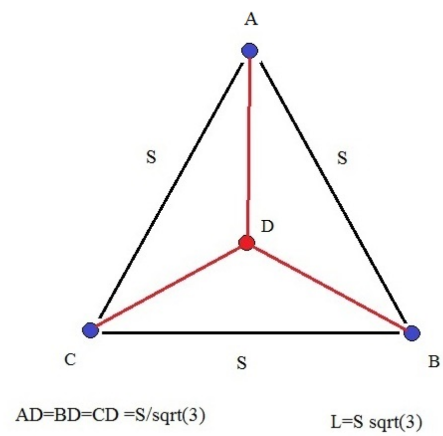
Consider next a three sink and one source configuration. We begin with the simplified case of three equally spaced sinks along the x axis designated by A[-1,0], B[0,0], and C[1,0] and one source at D[0,-3]. Here one would think a Steiner Fork connecting A,C and D plus a vertical line from B to the Steiner point at [0,-1.sqrt(3) should minimize the path. Doing the calculations it produces L=3+4/sqrt(3)=5.309. If, however, if we look at the alternate path L=(A-B)+(C-B)+(B-D)=1+1+3=5, we see that it is actually shorter. To prove this observation we consider an intersection point [0,y] of the four radial lines emanating from our source and three sinks. The length in this case is-

$$L=\sqrt{1+y^2}+y+\sqrt{1+y^2}+(3-y)=3+2\sqrt{1+y^2}$$

It has a minimum at y=0 and hence represents the T shaped path . The angles between the four radial lines become ± π/2 and ±π rad.

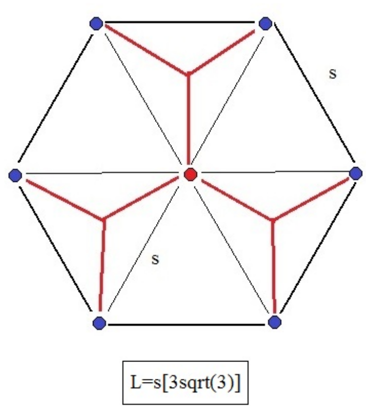
Another optimum path arises when we place three sinks at three of the corners of an equilateral triangle and a sink at it center. Superimposing two tilted Steiner Forks produces the minimum path shown in red in the following graph –

MINIMUM PATH BETWEEN THREE SYMMETRICALLY PLACED SINKS AND A CENTRAL SOURCE



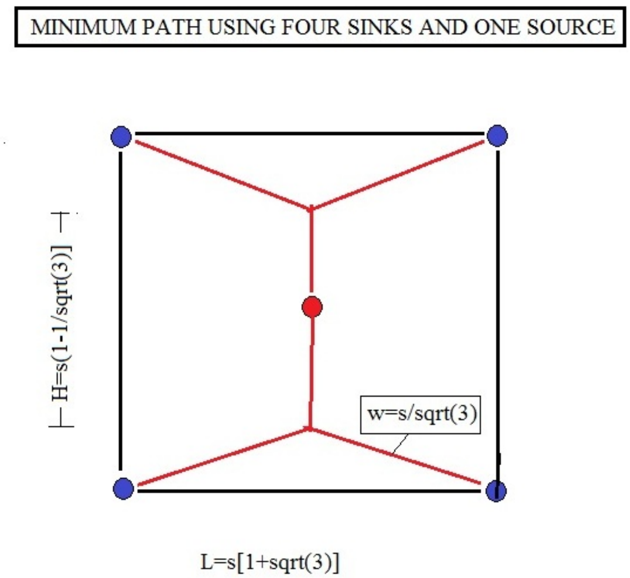
The optimum path shown in red has a length of any of the triangle sides multiplied by sqrt(3). These results may also be used to calculate the optimum path of six sinks sitting at the vertexes of a regular hexagon with one source at the origin. Here is the result obtained by simple superposition-

OPTMUM PATH (IN RED) CONNECTING SIX SINKS WITH ONE CENTRAL SOURCE



Each of the straight line segments have a length of Δ=s/sqrrt(3) so that the total path length becomes L=s[3 sqrt(3)]. Note that the pattern starts looking a bit like 2D soap-bubbles formed with the aide of dish washing material in a water filled sink liquids. This similarity should not be surprising since soap bubbles follow a minimum energy principle just as do paths discussed in the present problem. Soap bubbles take on a spherical shape for the same minimum energy reason.

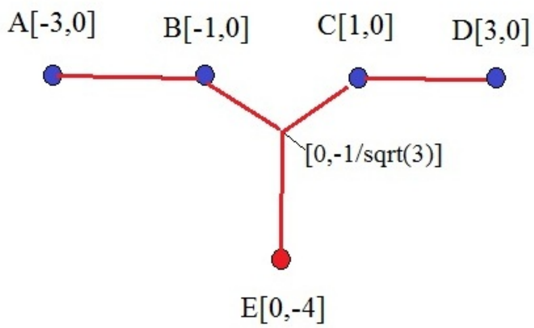
Let us continue on with a configuration where there are four sinks placed at the vertexes of a square of sides s. Plus there is one sink present at the square center. What will be the minimum path connecting all five points? Clearly we can again use Steiner Forks to solve the problem. We start by creating two Steiner Forks placed inside imaginary equilateral triangles inside the square as shown-



Although the handles of the two prong Steiner Forks are truncated, the angles between merging straight red lines remain at 120 deg. The length of this minimum path is L=s[1+sqrt(3)]=2.732s.If one were instead to run four radial diagonal lines from the square corners to the sink at the center, the length would have the larger value of L=4[s/sqrt(2)]=s[2 sqrt(2)]=2.828s.

We go next back to placing sinks along the x axis at regular intervals. Consider the four sinks located at [-3,0], [-1,0], [1,0], and [3,0] with a one sink at [0,-4] as shown-

OPTIMUM PATH CONNECTING FOUR LINE SINKS AND ONE SOURCE

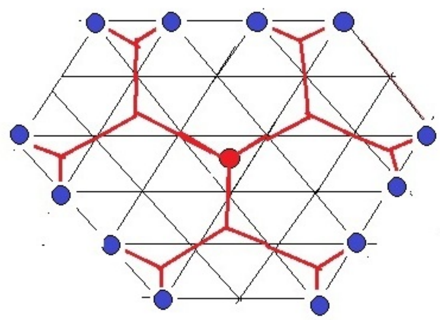


One Steiner Fork plus two horizontal lines
L=8+sqrt(3)

One sees at once that we can place one Steiner Fork with end coordinates [-1,0] , [1,0], and [0.-4]. Next considering placing additional forks gives no path advantage and one rather connects the remaining sinks with straight lines as indicated. The optimum path has length L=8+sqrt(3). Note that if an additional sink where to be placed at [0,0] the pattern would resort to the T form found earlier for three line sinks.

Finally let us see what patterns emerge if we specify one source and leave the number of sinks open bur specify that the path is to be a concatenation of classicasl Steiner Forks within equilateral triangles. The simplest way to draw such optimum paths is to first draw a triangular grid and then place a Steiner Fork into one of the triangles such that the end points of the fork coincide with the triangle vertexes. Continuing on to the next three triangles whose vertexes just touch the first triangle yields an extendd path. Continuing on we obtain the following picture-

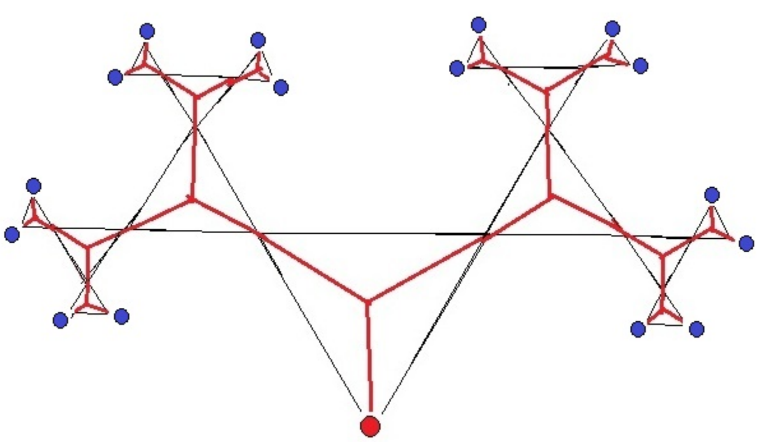
STEINER FORKS USED TO CONSTRUCT OPTIMUM DRAINAGE PATH INTO TWELVE SINKS FROM ONE CENTRAL WATER SOURCE



path constructed by use of equilateral triangular grid

The first three generations of Steiner forks yield the red line path to twelve sinks. If one took the concatenation an additional generation the prongs of the pattern would merge with its neighbor. The result of even more generations would yield a hexagonal pattern not unlike one encounters with 2D soap bubbles. One could avoid this overlap by making the forks placed into subsequent generations become smaller and smaller. In that case one would generate what is called a mathematical tree as shown-

CONSTRUCTION OF A FOUR GENERATION TREE USING DECREASING SIZE STEINER FORKS



It would be an interesting mathematical exercise to show that this configuration is indeed the minimum path between the source (shown as a solid red circle) and the 16 sinks (blue circles) at two of the vertexes of each of the fourth generation triangles.

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