VALUES OF THE SIGMA FUNCTION FOR LARGE SEMI-PRIMES

The sigma function \( \sigma(N) \) of number theory is defined as the sum of all divisors of \( N \). For semi-primes, where \( N = pq \) with \( p \) and \( q \) being two primes, it takes on the simpler form:

\[
\sigma(N) = 1 + pq + p + q
\]

Since most advanced math computer programs are able to recognize primes of over fifty digit length, one can use the above formula to find \( \sigma(N) \). Since \( p \) and \( q \) are odd numbers, with the exception of two, the sigma function will be an even number.

Let us demonstrate things beginning with the elementary semi-prime \( N = 77 = 7 \times 11 \). Here we get

\[
\sigma(77) = 1 + 77 + 7 + 11 = 96
\]

Most advanced mathematics computer programs have the built-in function \( \text{ithprime}(N) \) so that, for instance, the \( \text{ithprime}(46790) = 569663 = p \) and the \( \text{ithprime}(67912) = 854089 = q \). We thus have the semi-prime-

\[ N = pq = 486542902007 \]

So we get the even sigma function value-

\[
\sigma(N) = 1 + 486542902007 + 569663 + 854089 = 486544325760
\]

Our Maple program can handle this last result directly but will fail if the number of digits in \( N \) exceeds about fifteen. This does not mean, however, that the sigma function cannot be evaluated for any larger number \( N \) using the above formula. The reason for this is that the test for a number being prime pretty much works for numbers up to at least 100 digit length. So one needs to only find two large prime numbers \( p \) and \( q \) to generate a value for \( \sigma(N) \).

Let us demonstrate with the primes-

\[
q = 188495559215387594307758602996770173051830163962506349258496675538468984377172539917682089522091
\]

and-

\[
p = 163096909707542714121617248281159749865434825621997574498018057663444597821212855674282930711541
\]

Substituting into the equation we find-

\[
\sigma(N) = 1 + p \cdot q + p + q =
\]

\[
30743043201624841435668783130367668182125598728754413862766564068645223000
\]
Note that the dominant term in the sigma function formula is \( N=pq \). So we have the inequality-

\[
\sigma(N) > pq =
\]

30743043201624841435668783130367668182125598728754413862766564068645223000
166390972612299312186319057405838927934245787647805099913803703609034428
5261296971644755083300036850250867268152231

Note that the sigma function departs from \( N=pq \) by only a small fraction of \( \sigma(N) \). We also have for this last case that-

\[
\sigma(N)-pq = 1+p+q = 351592468922930308429375851277929922917264989584503923756514
733201913582198385395591965020233633
\]

So both \( \sigma(N)-pq \) and \( 1+p+q \) have the same value.

If one deals with large semi-primes \( N=pq \), where \( p \) and \( q \) are unknown primes, one can still find the sigma function value via the equalities-

\[
\sigma(N) = 1 + N + p + N/p \quad or \quad \sigma(N) = 1 + N + q + N/q
\]

Solving these for \( p \) and \( q \) we get-

\[
[p,q] = \left( \frac{1}{2} \right) \left\{ \sigma(N) - N - 1 \pm \sqrt{[\sigma(N) - N - 1]^2 - 4N} \right\}
\]

Since \( p \) and \( q \) must be integers, the term in the radical must equal the square of an unknown integer \( n \). Hence we have that-

\[
\sigma(N) = (1 + N) + \sqrt{4N + n^2}
\]

Also we find-

\[
p = \left( \frac{1}{2} \right) \left\{ \sqrt{4N + n^2} - n \right\} \quad and \quad q = \left( \frac{1}{2} \right) \left\{ \sqrt{4N + n^2} + n \right\}
\]

Let us evaluate the above results for some more semi-primes. We again start with the trivial case of \( N=77 \). Here \( \sigma(N) \), \( p \), and \( q \) demand that \( \sqrt{4(77)+n^2} \) be an integer. This is possible only for \( n=4 \) . So we have-

\[
[p,q] = 0.5 \{18 \pm 4\} = [7,11].
\]
and $\sigma$ equals-

$$\sigma(77) = 1 + 77 + 18 = 96$$

Consider next the larger semi-prime $N = 5207$. One finds that $n = 86$ produces the result –

$[p, q] = [41, 127]$ and $\sigma(5207) = 5376.$

You will notice, as expected, that $\sigma(N) > N$ in all cases. The difference between $\sigma(N)$ and $N$ is just $1 + p + q.$ But if we take $p < \sqrt{N}$ and $q > \sqrt{N}$ one gets-

$$(p + q) = \sqrt{[4N + n^2]} > 2\sqrt{N}$$

One can write $p = \alpha \sqrt{N}$ and $q = (1/\alpha) \sqrt{N}$, where $\alpha$ extends over the range $0 < \alpha < 1$. Thus it follows that-

$$(\alpha + \frac{1}{\alpha})\sqrt{N} = \sqrt{4N + n^2}$$

Hence one should commence the search with-

$$n = \{\sqrt{[\alpha + (1/\alpha)]^2 - 4}\} N^{1/2}$$

This result tells us that the computer search for the correct value of $n$ starts with a constant times the square root of $N$ and depends very much on the unknown size of $\alpha$. With a guess of $\alpha = 1/2$, one should start the search for the correct $n$ with $n = 1.5 \sqrt{N}$. On the other hand if $\alpha = 1/4$ we should commence the search at $n = 3.75 \sqrt{N}$.

To find the value for one last $\sigma(N)$, $p$ and $q$ consider the semi-prime –

$N = 455839$ where $\sqrt{N} = 675.158$

Evaluating $\sqrt{4N + n^2}$ using $\alpha = 1.1$ means we start with $n = 129$. The search program then reads-

```plaintext
for n from 129 to 129+50 do {n, evalf(sqrt(4*N+n^2))} od;
```

This produces the solution-

$$p + q = 1360 \quad \text{plus} \quad \sigma(N) = 457208 \quad \text{at} \quad n = 161$$

As a side benefit we also get $[p, q] = [599, 761]$.