SPHERICAL HARMONICS

When solving the Laplace equation in spherical coordinates one encounters the spherical Harmonics $Y_n^m(\theta, \phi)$. Let us look further into their properties and present graphs of some of these. Our starting point is the Laplace equation-

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin(\theta)} \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2(\theta)} \frac{\partial^2 V}{\partial \phi^2} = 0$$

Using the substitution $V = R(r)F(\theta)\exp(i m \phi)$ and the separation constant $n(n+1)$, leads to the solution-

$$R(r) = Ar^n + \frac{B}{r^{n+1}}$$

plus the second order ODE-

$$\sin(\theta)\frac{d}{d\theta} \left[ \sin(\theta) \frac{dF}{d\theta} \right] + \left[ n(n+1) \sin^2(\theta) - m^2 \right] F = 0$$

The substitution $x=\cos(\theta)$ allows this last equation to be rewritten as-

$$\frac{d}{dx} \left[ (1 - x^2) \frac{dF}{dx} \right] + \left[ n(n+1) - \frac{m^2}{(1 - x^2)} \right] F = 0$$

This equation has an exact solution in terms of associated Legendre polynomials and reads-

$$F = P_n^m(x) = (1 - x^2)^m \frac{d^m P_n(x)}{dx^m}$$

with $P_n(x)$ being the Legendre polynomial generated by the Rodriguez formula-

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

The Spherical Harmonics are then defined as-
\[ Y_n^m (\theta, \varphi) = F \exp(i m \varphi) = P_n^m (\cos \theta) [c_1 \sin (m \varphi) + c_2 \cos (m \varphi)] \]

where \( c_1 \) and \( c_2 \) are constants. The first few associated Legendre polynomials read-

\[
\begin{align*}
P[1,0] &= x \\
P[1,1] &= \sqrt{1-x^2} \\
P[2,0] &= \frac{1}{2} (3x^2 - 1) \\
P[2,1] &= 3x \sqrt{1-x^2} \\
P[2,2] &= 3(1-x^2) \\
P[3,0] &= \frac{x}{2} (5x^2 - 3) \\
P[3,1] &= \sqrt{1-x^2} \frac{3}{2} (5x^2 - 1) \\
P[3,2] &= (1-x^2)(15x) \\
P[3,3] &= 15(1-x^2)^{1.5} \\
P[4,0] &= \frac{1}{8} (35x^4 - 30x^2 + 3) \\
P[4,1] &= \sqrt{1-x^2} \frac{5x}{2} (7x^2 - 3) \\
P[4,2] &= (1-x^2) \frac{15}{2} (7x^2 - 1) \\
P[4,3] &= (1-x^2)^{1.5} (105x) \\
P[4,4] &= 105(1-x^2)^2
\end{align*}
\]

We have used \( P[n,m] \equiv P_{nm} \) and note that \( P[n,0] \) are just the standard Legendre polynomials and that \( P[n,n+k] = 0 \) when \( k=1,2,3,\ldots \). Reading from this list, we see for example that-

\[ Y_4^3 (\theta, \varphi) = 105 \cos (\theta) \sin (\theta)^3 \cos (3 \varphi) \]

A plot of this function looks like this-
This and other of the spherical harmonics produce interesting graphs which could be used to construct even more intricate patterns of possible graphic and artistic interest. The functions are orthogonal and hence may be used to represent functional shapes on the surface of a sphere. One has that-

\[ f(\theta, \varphi) = \sum_{n,m=0}^{\infty} c_{n,m} Y_n^m (\theta, \varphi) \]

and the coefficients \( c_{n,m} \) have value-

\[ c_{n,m} = \frac{\iint f(\theta, \varphi) Y_n^m (\theta, \varphi) dS}{\iint [Y_n^m (\theta, \varphi)]^2 dS} \]

With the surface area increment on a unit radius sphere given as \( dS = \sin(\theta) d\theta d\varphi \). Consider the function \( f(\theta, \varphi) = +1 \) for \( 0 < \theta < \pi/2 \) and \( f = -1 \) for \( \pi/2 < \theta < \pi \) which is independent of the azimuthal angle \( \varphi \). In terms of the earth, the function would have a value of one everywhere in the northern hemisphere and value minus one everywhere in the southern hemisphere. Here one finds-
\[
\begin{align*}
\int_{\theta=0}^{\pi/2} P_n(\cos\theta)\sin\theta \, d\theta - \int_{\theta=\pi/2}^{\pi} P_n(\cos\theta)\sin\theta \, d\theta \\
c_{n,0} = \frac{\int_{\theta=0}^{\pi} [(P_n(\cos\theta))^2 \sin\theta \, d\theta]}{\theta=0}
\end{align*}
\]

which can be rewritten as-

\[
\begin{align*}
1 \int_{x=0}^{1} P_n(x) \, dx - 0 \int_{x=-1}^{0} P_n(x) \, dx \\
c_{n,0} = \frac{1}{1} \frac{1 \int_{x=-1}^{0} [P_n(x)]^2 \, dx}{x=0} = (4n + 3) \frac{1 \int_{x=0}^{1} P_{2n+1}(x) \, dx}{x=0}
\end{align*}
\]

Thus the function \( f(\theta) \) can be written out as-

\[
f(\theta) = \frac{3}{2} P_1(\cos\theta) - \frac{7}{8} P_3(\cos\theta) + \frac{11}{16} P_5(\cos\theta) - \frac{75}{128} P_7(\cos\theta) + \frac{133}{256} P_9(\cos\theta) - \frac{483}{1024} P_{11}(\cos\theta) + \frac{891}{2048} P_{13}(\cos\theta) - \ldots
\]

We show you here in red the function found when summing up these first seven terms of an infinite series-

Seven Term Series Approximation for \( f(x) \)
This result clearly indicates that the series indeed approaches the values of +1 for 0<x<1 and -1 for -1<x<0. The usual Gibbs phenomenon at the function discontinuity at x=0 is also present.

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