# STICK CONSTRUCTION FOR ANY POLYGON

## INTRODUCTION:

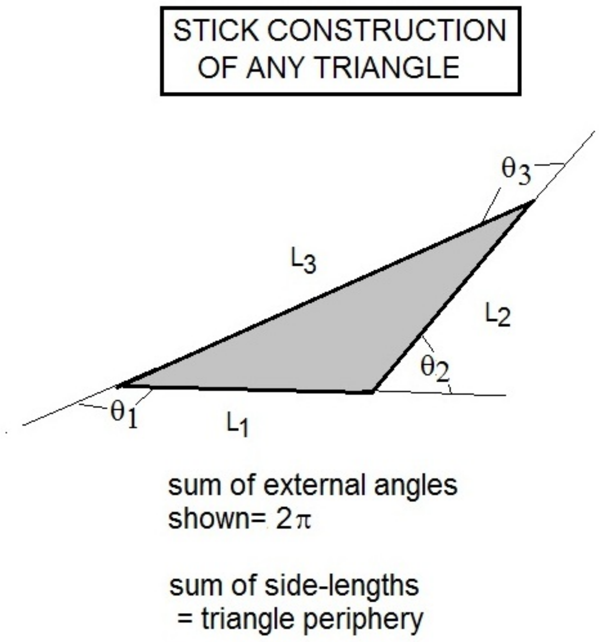
A polygon is any closed multi-sided 2D structure composed of N straight line elements connected to the endpoints of its neighbors. The simplest polygons are the triangle, quadrangle and hexagon. These structures can be thought of as a concatenation of sticks of specified length and given external angle. We want here to show how such stick arrangements can produce any closed polygon.

## STICK LENGTH AND ORIENTATION:

We begin our discussion by looking at a single stick of length $L_1$ lying along the x axis in the x-y plane. At its right end we place a second stick of length $L_2$ tilted at angle $\theta_2$ relative to $L_1$. The angle is considered positive if going counter-clockwise and negative if clockwise. A third stick of length $L_3$ is attached to the head of the second stick and makes an angle $\theta_3$ relative to $L_2$.Contimuing the process to additional sticks, it always becomes possible for the nth stick to hit the left end of stick one to form a closed figure of a polygon. The sum of the external angles will need to equal $2\pi$ and the sum of both the x and y components of all N sicks will need to add up to zero. It is our purpose here to construct several different polygons both regular and irregular. As we will see the present way of describing polygons often offers some interesting new formulas for determining the polygon areas.

## CONSTRUCTION AND FORMULAS FOR ANY TRIANGLE:

Let us begin the stick construction approach for the simplest of polygons, namely, the triangle. Here we have the following picture-




To produce this polygon requires first of all that the sum of the three external angles indicated sum to 2π radians. Also one has that the x and y components of the side lengths add up to zero. That is-

$$\sum_{n=1}^{3}\theta_n = 2\pi \qquad L_1 + L_2\cos(\theta_2) = -L_3\cos(\theta_1) \quad \textit{and} \quad L_2\sin(\theta_2) = L_3\sin(\theta_1)$$

Also we have that the area of this triangle is-

$$A_{triangle} = \frac{L_1 L_2}{2}\sin(\theta_2)$$

On using the standard Law of Sines and Law of Cosines learned in elementary trigonometry class, one finds-

$$\frac{L_2}{\sin(\theta_1)} = \frac{L_3}{\sin(\theta_2)} = \frac{L_1}{\sin(\theta_3)}$$

*and*

$$L_3^{\,2} = L_1^{\,2} + L_2^{\,2} + L_1 L_2\cos(\theta_2)$$

Consider now the area of an oblique triangle of side-lemgths $L_1=2$, $L_2=3$, and $L_3=4$. Here the above formulas yield-

$$\cos(\theta_2)=\frac{(16-4-9)}{2\cdot 6}=\frac{1}{4}\quad so\quad \sin(\theta_2)=\frac{\sqrt{15}}{4}\quad and\quad A_{triangle}=\frac{3}{4}\sqrt{15}$$

This is certainly a faster way to find a triangle area than by the use of the standard Heron formula. Also if we consider the sum of the internal angles of the triangle one gets-

$$(\pi-\theta_1)+(\pi-\theta_2)+(\pi-\theta_3)=3\pi-2\pi=\pi$$

This of course is the well known result that the sum of the three internal angles of any triangle always add up to π.

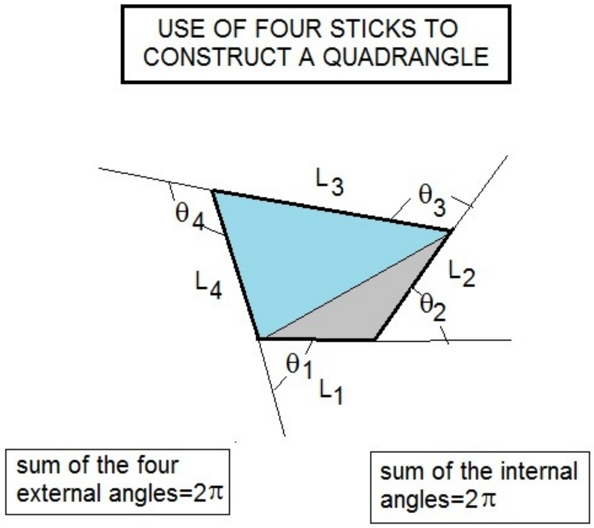
For this 2-3-4 triangle we find-

$$\arcsin(\theta_1)=-\frac{3\sqrt{15}}{16}\quad ,\quad \arcsin(\theta_2)=\frac{\sqrt{15}}{4}\quad and\quad \arcsin(\theta_3)=-\frac{\sqrt{15}}{8}$$

The sum of these three angles add up to precisely 2π radians.

**STICK CONSTRUCTION OF A QUADRANGLE:**

If we take four sticks of lengths $L_1$, $L_2$, $L_3$, $L_4$ and place them end to end so that they form a closed area we have generated a quadrangle as shown-




Because it is a closed figure the sum of the external angles shown must equal $2\pi$ and the sum of the four side-lengths will be the perimeter of the quadrangle. Three sides ($L_1$,$L_2$ and $L_4$) and two angles($\theta_1$,$\theta_2$ ) are sufficient to find the length of the remaining stick $L_3$ . We find-

$$L_3 = \sqrt{[L_1 + L_2\cos(\theta_2) + L_4\cos(\theta_1)]^2 + [L_2\sin(\theta_2) - L_4\sin(\theta_1)]^2}$$

In addition the diagonal line d inside the quadrangle has the known length-

$$d = \sqrt{L_1^{\ 2} + L_2^{\ 2} + 2L_1L_2\cos(\theta_2)}$$

by the Law of Cosines. The area of the quadrilateral can be determined by adding the areas of the two triangles lying in the quadrilateral which are separated by d. Using either a vector approach or Heron's Formula one can find the area of each of the sub-triangles to produce the total area contained within the quadrangle.

Let us demonstrate the area determination procedure for the specific quadrilateral of side-lengths $L_1$=2, $L_2$=3, and $L_4$=4 with angles $\theta_1=\pi/4$ and $\theta_2=\pi/6$. For these conditions we find d=sqrt(7) and-

$$L_3^{\ 2} = (2 + 3\sqrt{3}/2 + 4/\sqrt{2})^2 + (3/2 - 4/\sqrt{2})^2 = 56.917$$

Thus $L_3$=7.544. To get the total area contained in this quadrangle we use the vector approach. Three of the side-length vectors and the d vector are-

$$\vec{L}_1 = -2i$$

$$\vec{L}_2 = \frac{3}{2}(\sqrt{3}i + j)$$

$$\vec{L}_4 = \frac{4}{\sqrt{2}}(-i + j)$$

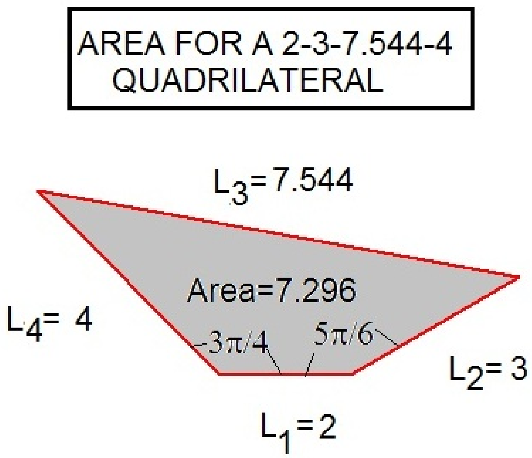
*and*

$$\vec{d} = \frac{1}{2}[(4+3\sqrt{3})\,i + 3\,j]$$

Using the cross product in vector multiplication we then have-

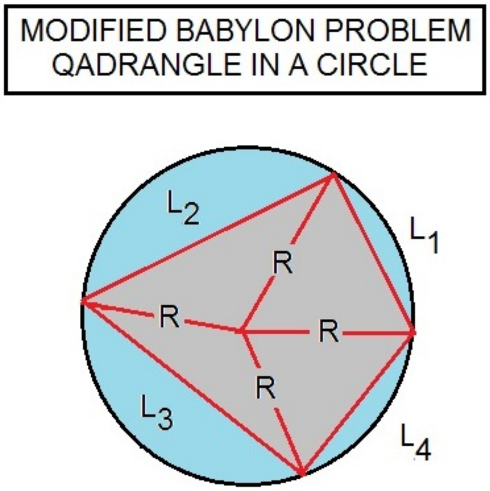
$$A = A_1 + A_2 = \frac{3}{2} + \frac{3}{\sqrt{2}}[1+\sqrt{3}] = 7.2955$$

This way of finding the total area is a little faster than using the Heron formula. The following shows a picture of this particular quadrangle-




We point out that the simplest quadrangle is the square where the external angles all have value $\theta=\pi/2$ and all sides are of equal length L.

Another interesting problem involving stick construction is that of the area of a quadrangle lying inside a circle of radius R. This can be thought of as a modification of the famous Babylonian problem of a triangle in a circle dating back to about 2000BC. Here we have the configuration shown-




To get the total area of the quadrangle with sides $L_1$, $L_2$, $L_3$, and $L_4$ one simply adds up the area of the four isosceles triangles shown. That is-

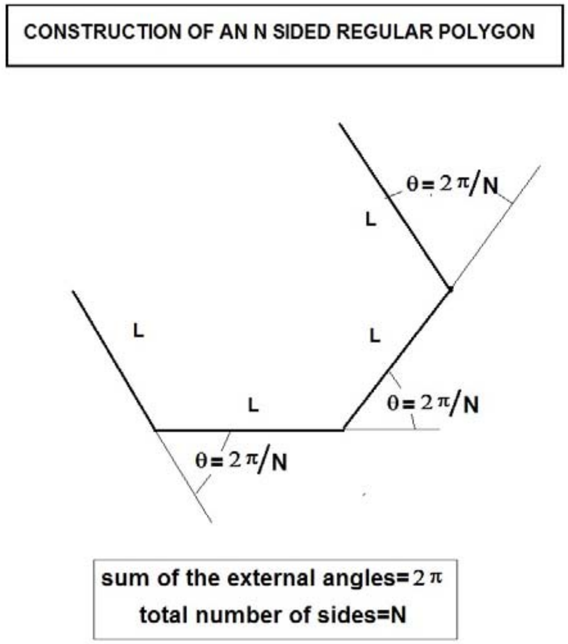
$$A_{quad} = \frac{R}{2}\sum_{n=1}^{4} L_n \cos(\frac{\varphi_n}{2})$$

Here $\varphi_n$ is the angle subtended at the circle center and the radial lines going to the nth and (n+1) vertex. In the case where all the Ls are equal and the angles $\varphi_n$ are equal to $\pi/2$, we get the area of a square with side-length L. One finds-

$$A_{square} = 2RL\cos(\frac{\pi}{4}) = L^2$$

**HIGHER ORDER REGULAR POLYGONS:**

We next want to look at the stick construction of higher order regular polygons. These regular polygons are created by the concatenation of N equal length sticks of length L laid end to end with an external bend angle of $\theta=2\pi/N$ relative to each other. Here is a schematic-




Continuing laying sticks in this manner will form a closed irregular polygon after N sticks have been laid. One gets an equilateral triangle, a square, a pentagon, a hexagon, and an octagon using N=3, 4, 5, 6, and 8, respectively.

The perimeter of these polygons is NL. To calculate the area one first draws N straight lines from the polygon center to each vertex and then determines the area of each of the resultant pie shaped section and multiplies this area by N. The resultant total area follows the simple formula-

$$A_{poly} = N\frac{L^2}{4}\cot(\frac{\pi}{N})$$

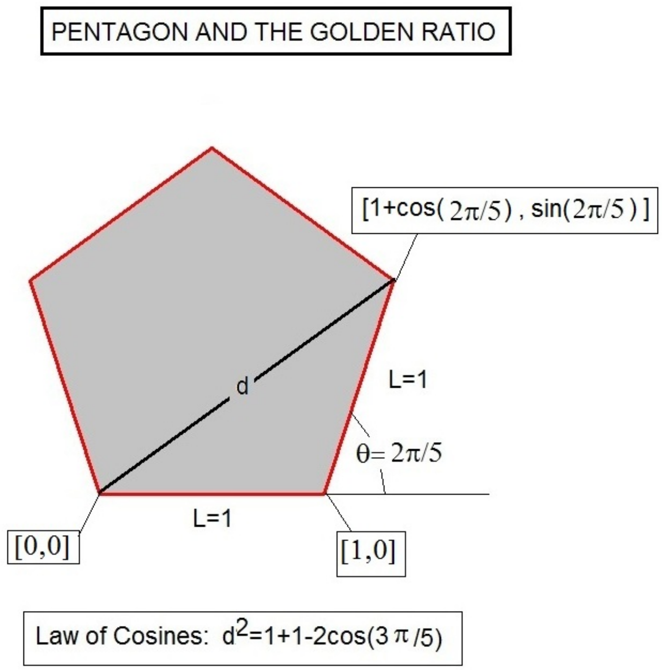
This means that a regular hexagon has a total area of $A_{hex}$=[3sqrt(3)/2]$L^2$.
An octagon has $A_{oct}$=2[1+sqrt(2)]$L^2$. A twenty sided polygon yields-

$$A_{20} = 5L^2/\tan(\frac{\pi}{20}) \approx \frac{100}{\pi}L^2$$

Note that the $A_{20}$ area lies very close to the area of a circle. Archimedes was the first person to use this fact to obtain a good estimate for π using both inscribed and circumscribed polygons with large N.

For a regular pentagon , the external angle equals θ=2π /5 rad=72deg. When L=1 this polygon has the interesting property that its diagonal d just equals the Golden

Rratio R=[1+sqrt(5)]/2=1.680…. To prove this observation we start with the following graph-




The length of the d line follows from-

$$d^2 = [1+\cos(\frac{2\pi}{5})]^2 + [\sin(\frac{2\pi}{5})]^2$$

$$= 2[1+\sin(\frac{\pi}{10})] = (1.6180..)^2$$

So we have-

$$d = \frac{[1+\sqrt{5}]}{2} = 1.6180339887498948482045868343656381\ldots$$

which is the Golden Ratio .

**CONCLUDING REMARKS:**

We have shown that one can construct any N sided polygon ( be it regular or irregular) by the concatenation of sticks so that neighboring sticks make a specified external angle with respect to each other. For the resultant figure to be a closed polygon requires that the sum of the external angles add up to 2π radians and also that both the x and y components of all side-lengths add up to zero. If these three conditions are not met, then the resultant figure will be an open figure such as a spiral, staircase function, etc, , One can always calculate the areas of any polygon once most of the lengths L and orientation angles θ have been specified.

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