# CONSTRUCTION OF SUDOKU SQUARES

In the last few weeks I have been working out 9 x 9 Sudoku puzzles as they appear daily in our local newspaper. Although I am becoming fairly proficient in solving these, it still takes me at least thirty minutes to complete the typical 81 element Sudoku Puzzle. In one sense such efforts can be considered a waste of time since the time could be better spent on other endeavors. Fortunately, during these efforts (which can become addictive) I have come up with a new approach of looking at Sudoku problems in reverse. That is, we first generate a complete generic Sudoku Square, examine its general properties, and then use it to quickly generate different Sudoku puzzles. As will be seen, this reverse procedure is much faster then the standard solution process.

To generate Sudoku Squares we begin with a simple 4 x 4 case using just the four letters A, B, C, and D as our elements. We designate the elements in the first row as A-B-C-D and the first column by A-C-B-D and then proceed to write the remaining elements using the four letters in cyclic fashion. The resultant generic square reads-

$$S = \begin{vmatrix} A & B & C & D \\ C & D & A & B \\ B & C & D & A \\ D & A & B & C \end{vmatrix}$$

As required, all rows and columns contain the base elements A, B, C, and D just once, as is also the case for the four sub-matrixes. In any Sudoku Square of $n^2$ total elements there will always be n sub-matrixes each containing the base letters just once. Thus for the present case of n=4, there are four sub-matrixes. In looking at this matrix one notices that there are numerous related configurations allowed which will not destroy its basic properties. That is, both the two rows and columns passing through each sub-matrix can be interchanged and four rotations of 90 degrees each of the entire matrix are allowed. For example, columns 3 and 4 can be interchanged, followed by an interchange of rows 1 and 2, and ending with a rotation of the entire square by 90 degrees counter-clockwise. The resultant Sudoku Square has the form-

$$S = \begin{vmatrix} A & C & D & B \\ B & D & A & C \\ D & B & C & A \\ C & A & B & D \end{vmatrix}$$

It retains the properties of the original Sudoku Square but clearly the elements of the matrix have been scrambled. It is also possible to introduce even more scrambling by relaxing the cyclic placement form of the letters used in generating the remaining rows and columns.

What is clear from these results is that generic Sudoku Squares are numerous with rapid increases in their number expected as n increases This fact makes the solution of Sudoku Puzzles ever more difficult as n is increased and is the reason why most Sudoku Puzzles do not consider squares of 256 and 625 elements or larger.

The above two generic 4 x 4 Sudoku Squares represent two of many other possible configurations. Each of these completed squares have associated with it numerous Sudoku Puzzles. We demonstrate this by looking at the second generic form given above after setting A=1, B=2, C=3, and D=4. This produces the square matrix-

$$\begin{vmatrix} 1 & 3 & 4 & 2 \\ 2 & 4 & 1 & 3 \\ 4 & 2 & 3 & 1 \\ 3 & 1 & 2 & 4 \end{vmatrix}$$

From this solution one can write down , without much effort, several different Sudoku Puzzles including-

$$\begin{vmatrix} 1 & - & - & - \\ - & 4 & - & - \\ - & - & - & 1 \\ - & - & 2 & - \end{vmatrix} \quad and \quad \begin{vmatrix} - & - & 4 & - \\ - & - & - & 3 \\ - & 2 & - & - \\ - & - & 1 & - \end{vmatrix}$$

In each case one is being asked to find the remaining 12 elements indicated by the dashes. For such 4 x 4 puzzles the solution is an easy task but it will not be when n becomes 9, 16, or larger.

We next consider 9 x 9 generic Sudoku Squares which contains among its numerous configurations the solution to all standard 81 element Sudoku Puzzles . The simplest way to generate such squares is to write the first row of the matrix as A-B-C-D-E-F-G-H-I and then shift this sequence in subsequent rows but keeping the lettering in order. This results in-

$$\begin{vmatrix} A & B & C & D & E & F & G & H & I \\ D & E & F & G & H & I & A & B & C \\ G & H & I & A & B & C & D & E & F \\ B & C & D & E & F & G & H & I & A \\ E & F & G & H & I & A & B & C & D \\ H & I & A & B & C & D & E & F & G \\ C & D & E & F & G & H & I & A & B \\ F & G & H & I & A & B & C & D & E \\ I & A & B & C & D & E & F & G & H \end{vmatrix}$$

You will note that each column, row, and 3 x 3 sub-matrix contains the nine letters just once as required for a Sudoku Square. Also it is clear that the three rows and three columns passing through each of the sub-matrixes can be interchanged with the properties of the Sudoku Square remaining intact. Multiple rotations of 90 degrees of the entire matrix also will leave the Sudoku properties unchanged. Thus we clearly have a huge number of configurations for 9 x 9 Sudoku Squares exceeding $6^6$x4=186624 when the cyclic requirement for letter listing is relaxed. This fact explains why Sudoku puzzle designers are unlikely to ever run out of solution configurations for constructing standard 81 element Sudoku Puzzles.

To show you one example of applying the reverse procedure to a 9 x 9 Sudoku problem, we look at a different configuration of our generic matrix. We scramble the matrix by interchanging row 1 with row 3, followed by interchanging column 7 with column 8, and then rotating things by 180 degrees counter-clockwise and finally setting A=1 B=2, C=3, D=4, E=5, F=6, G=7, H=8, an I=9. The resultant Sudoku Square reads-

$$\begin{vmatrix} 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 9 \\ 2 & 1 & 9 & 8 & 7 & 6 & 5 & 4 & 3 \\ 5 & 4 & 3 & 2 & 1 & 9 & 8 & 7 & 6 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 9 & 8 \\ 4 & 3 & 2 & 1 & 9 & 8 & 7 & 6 & 5 \\ 1 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 \\ 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 3 & 2 & 1 & 9 & 8 & 7 & 6 & 5 & 4 \\ 6 & 5 & 4 & 3 & 2 & 1 & 9 & 8 & 7 \end{vmatrix}$$

We can generate many different Sudoko puzzles from this last result, Typically to have a unique solution requires that one begins the 9 x 9 puzzle with at least 27 to 30 knowns . The remaining terms can then be found (often after considerable effort on the part of the solver).

Here is one of the many possible puzzles which has this last square as its solution-

$$\begin{vmatrix} 8 & - & - & 5 & - & - & 2 & - & 9 \\ - & 1 & - & - & - & - & 5 & 4 & - \\ - & 4 & 3 & 2 & - & 9 & 8 & - & - \\ - & 7 & 9 & - & - & - & 1 & - & - \\ - & - & - & - & - & - & 7 & 6 & - \\ 1 & - & - & - & - & - & - & 3 & - \\ - & - & - & - & 5 & 4 & 3 & - & 1 \\ - & - & 1 & - & - & - & 6 & - & - \\ - & - & 4 & 3 & 2 & - & 9 & 8 & - \end{vmatrix}$$

It has a total of 30 known at the beginning of the puzzle and should be of medium difficulty to solve. Without referring to the above generic matrix, it took me a little over 1 hour to come up with the complete solution.

From these results one can conclude that it is very easy to generate an n x n generic Sudoku Square, but difficult to actually solve a Sudoku Puzzle when n x n=81 or larger. The Sudoku Problem is an example of an asymmetric math problem, not unlike what one encounters when trying to factor large semi-primes where it is easy to multiply two primes together but very time consuming to factor the resultant semi-prime.