SUMMING INTEGERS TAKEN TO THE PTH POWER

If one looks at the first few integers taken to the first power each one gets the following array-

\[ S(1) = 1 \]
\[ S(2) = 1 + 2 = 3 \]
\[ S(3) = 1 + 2 + 3 = 6 \]
\[ S(4) = 1 + 2 + 3 + 4 = 10 \]

This means that \( S(n) - S(n-1) = n \) and \( S(n) - S(n-m) = mn - S(n-2) \). So if we set \( n-m=1=S(1) \), we get the summation formula-

\[ S(n) = n(1+n)/2 \]

This point function is valid for all positive integers and clearly goes to infinity as \( n \) approaches infinity. An alternate way to arrive at this quadratic equation result is to simply regroup the integers as \((1+n) + (2+n-1) + (3+n-2) + \ldots \) taken \( n/2 \) times. So the sum of all integers through \( n=100 \) equals 5050.

If we just want the sum only the all odd integers we get-

\[ S(1) = 1 \]
\[ S(3) = 1 + 3 = 4 \]
\[ S(5) = -1 + 3 + 5 = 9 \]
\[ S(7) = 1 + 3 + 5 + 7 = 16 \]

This at once says that –

\[ S(2n-1) = n^2 \]

by induction. So the sum of all odd integers through 19 is 100.

To arrive at the sum of all even numbers through \( 2n \) we have the array-

\[ S(2) = 2 \]
\[ S(4) = 6 \]
\[ S(6) = 12 \]
\[ S(8) = 20 \]

The fact that the second differences are equal to two in each case implies we have a quadratic of the form \( An^2 + Bn = S(2n) \), were \( A \) and \( B \) are to be determined using \( S(2) \) and \( S(4) \). We find \( A=B=1 \) to yield the general form-

\[ S(2n) = n(n+1) \]
So the sum of all even numbers through 20 is $S(20)=110$.

We next proceed on to the square of the first $n$ integers. This produces the array:

- $S(1)=1$
- $S(2)=1+4=5$
- $S(3)=1+4+9=14$
- $S(4)=1+4+9+16=30$
- $S(5)=1+4+9+16+25=55$

The first difference is here 4-9-16-25, the second difference is 5-7-9, and the third difference is 2-2. This implies that $S(n)$ is a cubic in $n$ of the form $An^3+Bn^2+Cn$. Using the first three results in the array we get $A=1/3$, $B=1/2$, $C=1/6$. Hence we have the general form:

$$S(n)=[n(n+1)(2n+1)]/6$$

So the sum of the squares of the first 50 integers is 42925.

We could go on to summing integers to powers 3, 4, 5 etc. What the summing of the first and second powers have shown is that the sum of integers taken to the $p$th power will be represented by polynomials whose highest power will be $p+1$. So, for example, the sum of the first $n$ integers taken to the third power will be the 4th order polynomial $S(N)=n^2(1+n)^2/4$.

Although the sum of all positive powers of the integers becomes infinite as $n$ goes to infinity, this will generally not be the case when taking the integer sum to negative powers. Although the harmonic series:

$$S(n)=1+1/2+1/3+1/4+\ldots$$

where the power is $p=-1$, does blow up as $n$ goes to infinity, when $p=-2,-3,-4\ldots$ etc we get finite values. Take for example the sum:

$$S(n)=1+1/4+1/9+1/16+1/25+\ldots+1/n^2=\sum_{k=1}^{n} \frac{1}{k^2}$$

Here $S(10)=1.54976$, $S(100)=1.63498$, and $S(1000)=1.64393$ when $n\to\infty$. Also our computer shows that $S(\text{infinity})=1.644934$. Leonard Euler was the first to show that this last irrational number equals exactly $\pi^2/6$. More generally those sums involving negative powers $p$ when taken to infinity represent the zeta function:
\[ \zeta(p) = \sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots \]

There are still a few unsolved problems associated with the zeta function. One of these is the yet to be proven Riemann Hypothesis which states that \( \zeta(\sigma+i\tau) \) has all of its zeros lie along the line \( \sigma=1/2 \) in the complex plane.

Variations on the zeta function can be produced by alternating signs in the series, Thus we can define a new function-

\[ N(p) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \ldots \]

A plot of this function and its derivative follows-

The function goes to \( N(0)=0.5 \) and \( N(\infty)=1 \).

When \( p=1 \) this function has the form-

\[ N(1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \frac{\pi}{4} \]

This series is notoriously slow to converge to a finite value while \( \zeta(1) \) has infinite value. If we take \( p=2 \), we get the series-
\[ N(2) = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \ldots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = 0.82246703342411321824 = \frac{\pi^2}{12} \]

This means that \( \zeta(2) = 2N(2) \).

The differentiation of \( N(p) \) with respect to \( p \) also produces the result-

\[ \sum_{k=1}^{\infty} \frac{(-1)^k \ln(k)}{x^p} = \frac{\ln(2)}{2^p} - \frac{\ln(3)}{3^p} + \frac{\ln(4)}{4^p} + \ldots \]

This last result at \( p = 1 \) means that-

\[ \frac{\ln(2)}{2} - \frac{\ln(3)}{3} + \frac{\ln(4)}{4} - \frac{\ln(5)}{5} + \ldots = 0.1598689037 \]

If we retain only the odd terms in \( N(p) \) one gets an additional function-

\[ M(p) = 1 - \frac{1}{3^p} + \frac{1}{5^p} - \frac{1}{7^p} + \ldots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^p} \]

For the special case of \( p = 2 \) one finds \( M(2) \) to be equal to the -

Catalan Constant \( = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^2} = 0.91596559417721901505.. \)

Also \( M(1) \) yields the Gregory series-

\[ M(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4} \]

Upon differentiating \( M(p) \) with respect to \( p \) we find-

\[ \frac{dM(p)}{dp} = \sum_{k=1}^{\infty} \frac{(-1)^k \ln(2k-1)}{(2k-1)^p} = \frac{\ln(3)}{3^p} - \frac{\ln(5)}{5^p} + \frac{\ln(7)}{7^p} - \ldots \]

This leads to the identity-

\[ \sum_{k=1}^{\infty} \frac{(-1)^k \ln(2k-1)}{2k-1} = \frac{\ln(3)}{3} - \frac{\ln(5)}{5} + \frac{\ln(7)}{7} - \frac{\ln(9)}{9} + \ldots = 0.1929013168 \]
This is again a very slowly converging series to the value shown.

There is one more things one may do with the sums given above. One can always look at products of these series. Consider for example the product-

\[ P(n,m) = (1+2+3+\ldots+n)(1+2+3+\ldots+m) = \frac{nm(1+n)(1+m)}{4} \]

So if \( n=10 \) and \( m=5 \) we get \( P(10,5) = \frac{50(11)(6)}{4} = 825 \). Also we get-

\[ \lim_{n \to \infty} \left( 1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2} \right)^2 = \frac{\pi^4}{36} = 2.705808084277845\ldots \]

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