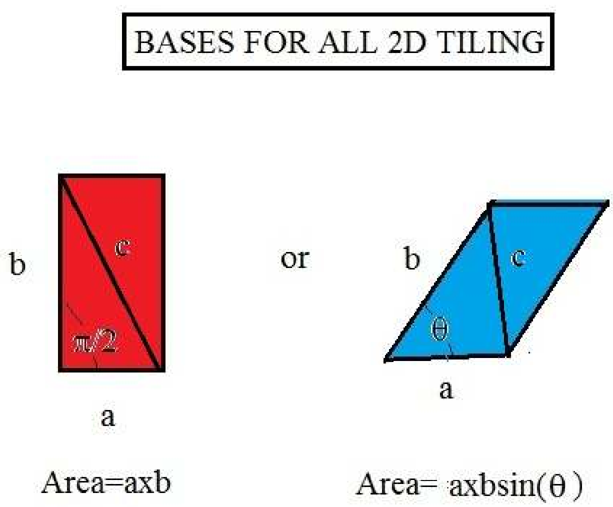
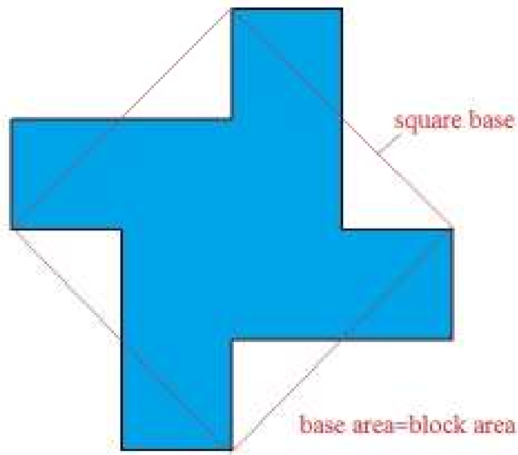
TILING IN THE PLANE

Recently while repairing some tiles in my kitchen, I got to thinking more about what are all possible shapes of tiles which can cover a flat surface without leaving gaps between the tiles. The answer is a bit more complicated than first thought. Essentially one starts with the simplest shapes namely rectangles of height ‘b’ and width ‘a’ of which squares are special cases. One can also consider the two triangles formed by a diagonal cut through the rectangular tile as a tile bases. In addition, one can have oblique tiles where the angle between side ‘a’ and neighboring side’ b’ is equal to θ. In that case the tile will have the shape of a rhomboid as indicated. In addition one has that such oblique tiles are equivalent to two identical triangles of sides a, b, and c. Here is a picture of the tile bases-



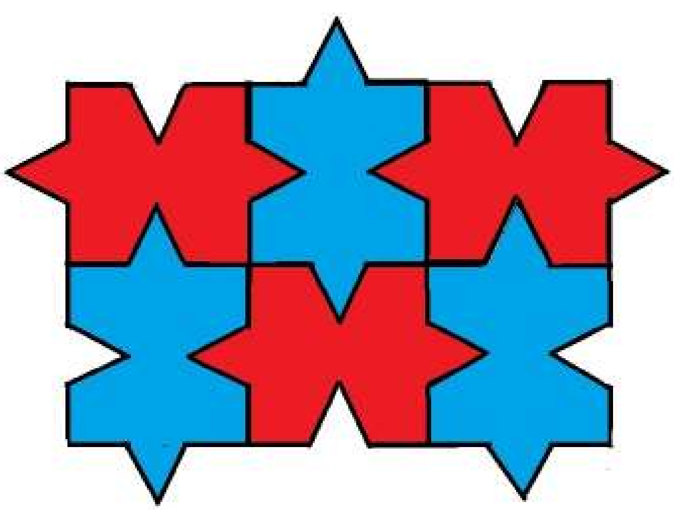
Clearly both of these types of tiles will cover any plane surface without producing gaps. However, there are many more. For example if we look at a unit side square as a tile base and then distorts the square sides as indicated -

POSSIBLE BUILDING BLOCK FOR CONTIGUOUS 2D TILING



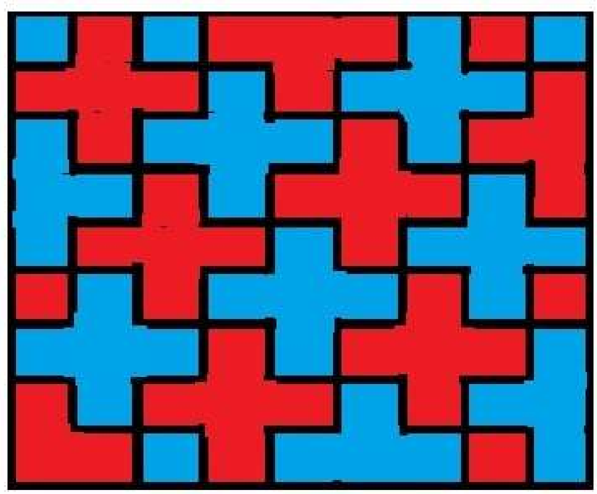
one obtains the blue configuration. This figure is constructed by bisecting the square sides with a positive and negative right triangle such that the area of the blue tile remains the same as the original square. Clearly a collection of such tiles will also cover a plane entirely. There are an infinite number of ways to conserve area in a distortion of the square sides and thus one has essentially an infinite number of different shapes capable of covering a plane. We show you here another example where this time each side is distorted by isosceles triangles-

CONTIGUOUS TILING PATTERN PRODUCED BY ISOCELES TRIANGLE PROTRUSIONS AND INDENTATIONS ON A SQUARE BASE



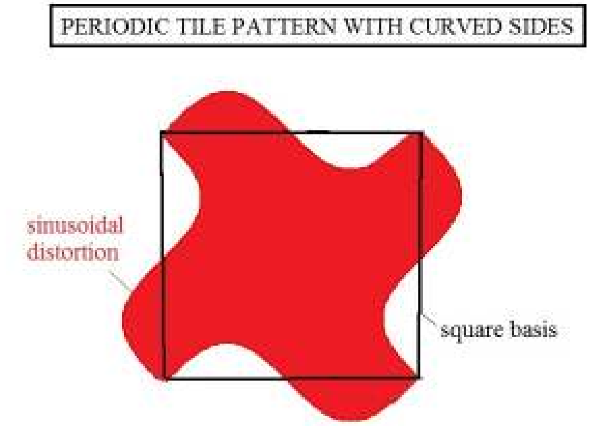
Notice that tiles in the shape of a standard Swiss Cross or in the form of an equilateral triangle can also be used to produce no-gap tile arrays. For the Swiss cross we use a different approach of rotating the cross in ninety degree increments about one of its outer corners. Here is the result-

TILING OF A PLANE WITH SWISS CROSSES



These last two patterns are very reminiscent of early Islamic Tile art. One can view similar patterns at the Topkapi Palace in Instanbul and the Alhambra in Granada, Spain. They also remind one of some of the works of the Dutch artist M.C.Escher.

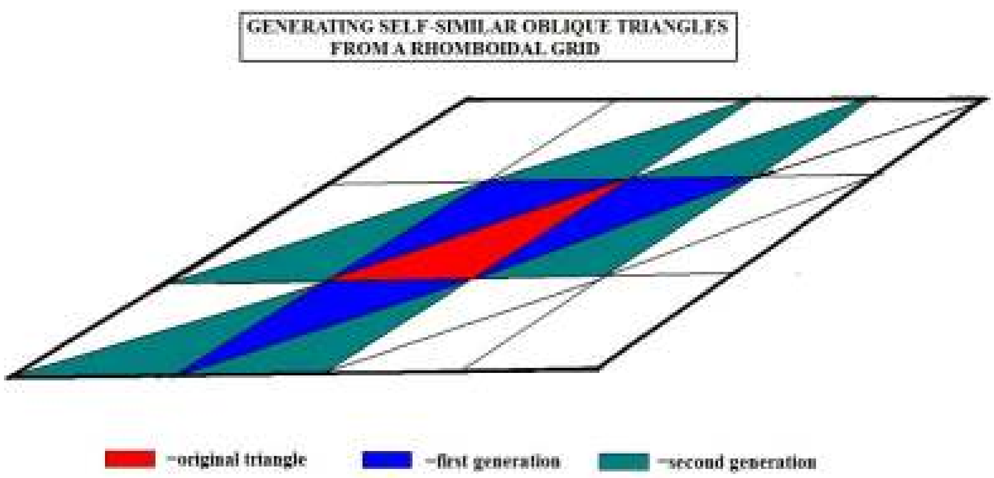
We want to point out that it is not necessary that a tile have only straight edges. Other shapes with curved edges can readily be constructed. As an example, consider placing a sinusoidal displacement onto each edge of a square. The result will be the following-



Area of tile and square are equal

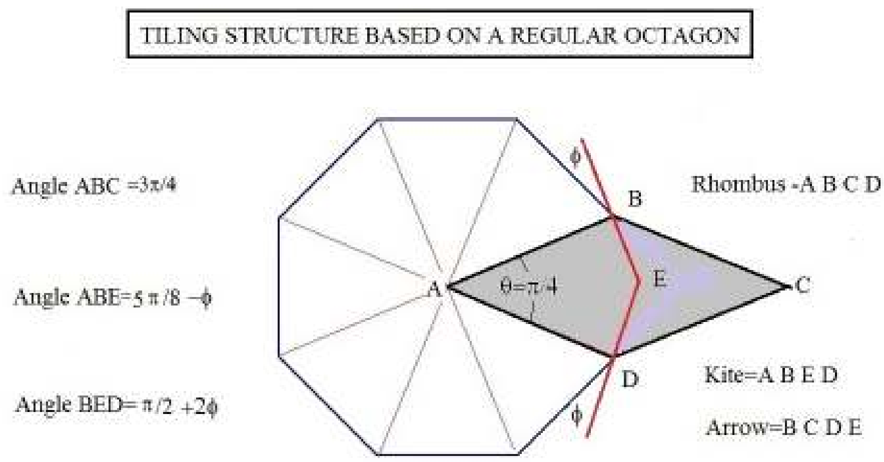
Again this type of tile was anticipated over 800 year ago by Islamic artists .

So far we have only dealt with tilings involving a square base. We now extend things to oblique rhomboidal bases. As the first example we consider an oblique triangle ABC and bring together its longest sides to form a base rhomboid. It shows at once that one can, without distorting the sides of the triangle, produce a tiling with no gaps. Here is a picture of such a tiling-

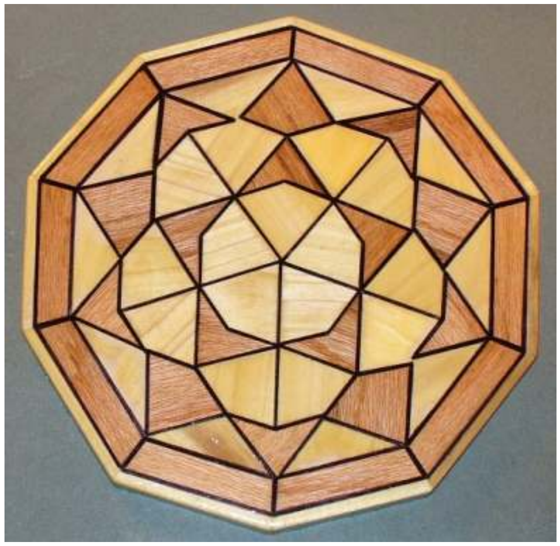


Many people are under the impression that such a tiling without gaps is impossible, but this should no longer be the case after looking at this last figure.

A convenient way to consider tilings based on oblique bases is to use regular polygons to generate the base rhombus. The next figure demonstrates the procedure for a regular octagon-

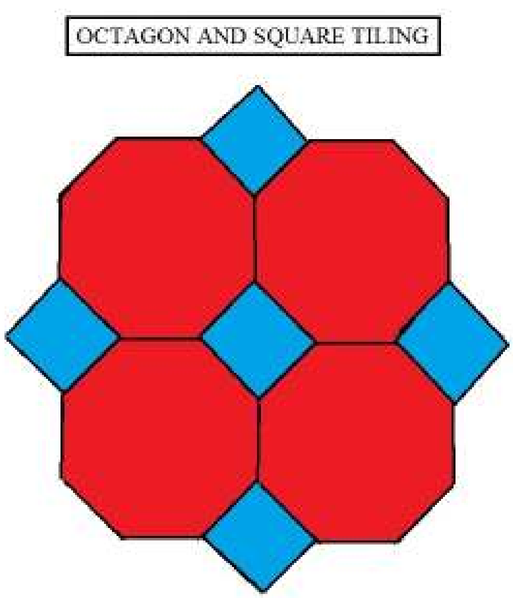


The base rhombus ABCD is shown in gray. By the nature of the regular polygon all four of its sides have the same length(and hence it should be referred to as a rhombus and not a rhomboid) with the acute angle equal to π/4 radians and the obtuse angle equal to 3π/4 radians. As with the square or rectangle base, we can distort the sides of this rhombus in any manner which conserves the original area. Doing so will produce a contiguous tiling covering the entire flat plane. It is also possible to break the rhombus up into a kite and arrow area as indicated. These can be combined in various ways to produce interesting two tile tilings. For the case where the polygon generator is a pentagon, one finds special forms of the kite and arrow which lead to Penrose tiling. These tilings are interesting because they produce a contiguous but not quite periodic tiling pattern. A wood inlay plate which I constructed a couple of years ago demonstrating Penrose tiling looks like this-

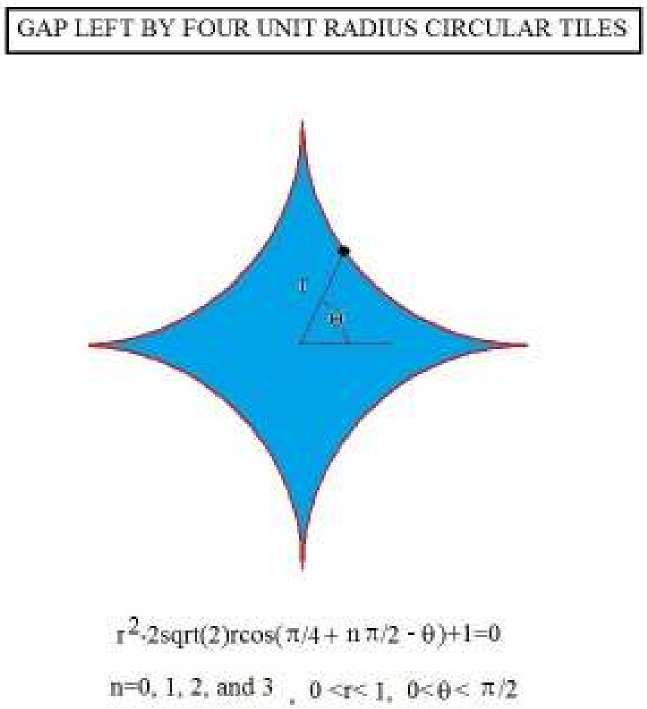


One of the simplest oblique tilings is associated with the hexagon. There the base rhombus has an acute angle of $\pi/3$ and an obtuse angle of $2\pi/3$. This rhombus can be broken up into two identical equilateral triangles. These equilateral triangles can then be used to produce hexagonal tiles which cover the entire plane without gaps.

All regular polygons, with the exception of the equilateral triangle, the square, and the hexagon, will not cover a plane without leaving gaps. However these gaps typically have the form of simple smaller areas which can be covered with a second set of tiles allowing complete coverage. For the case of the octagon one is left with gaps in the form of small squares as indicated-



Thus this combination of octagons and squares produce a tiling array without gaps. If one goes to the limit of n equal to infinity the larger areas become circles with the gap reducing to the scalloped area shown-



We were dealing with gaps like this and related three cusps areas in earlier studies on oscillatory heat transfer.

Another two tile tiling with curved boundaries can be constructed using equilateral triangles as a base. If one takes such a triangle and draws three circular arcs centered on the third corner and passing through the opposite two corners, there will result the well known Reuleaux Triangle. It and its complement can fill the plane as indicated-

TWO-TILE TILING BASED ON REULEAUX TRIANGLES
(base pattern is an equilateral triangle)