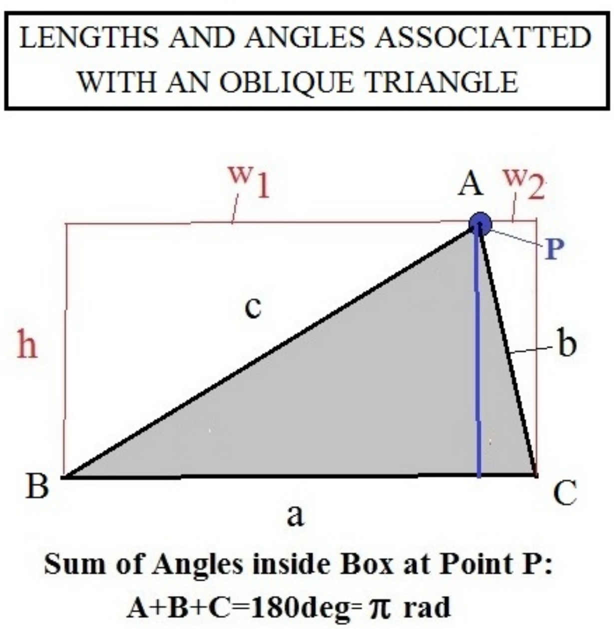
A LOOK AT THE PROPERTIES OF TRIANGLES

The simplest two dimensional figure is the triangle. It is constructed by the intersection of three non-parallel lines. Students first run into this figure in early grade school and learn about its more esoteric properties by the time they graduate from high school. In college majors in mathematics, physics and engineering make extensive use of its properties. Our purpose here is to quickly derive all its properties using just elementary algebra and geometry. The order of the discussion will differ somewhat from the standard sequential methods used in most mathematics curricula.

Area of Any Triangle:

Our starting point is to draw a simple rectangle of width w and height h. We locate a point P along the top of this rectangle and from it draw two straight lines of length b and c to the opposite bottom corners of the rectangle as shown-



This defines the shaded oblique triangle of side-lengths are a, b, and c and its three vertex angles designated by A, B, and C. A vertical blue line breaks the oblique triangle into two right triangles. The base length of the oblique triangle equals $a = w_1 + w_2$, so that from the Pythagorean Theorem one has –

$$a = \sqrt{c^2 - h^2} + \sqrt{b^2 - h^2}$$

We see that the area of the oblique triangle is just the area of the rectangle minus the areas of the two right triangles lying outside the triangle. We thus have

Area ObliqueTriangle= $h(w_1 + w_2) - (\frac{h}{2})(w_1 + w_2) = (\frac{ah}{2})$

This is a well known result. We can eliminate the rectangle height h via the easily established identity-

$$h = \frac{1}{2a}\sqrt{(2cb)^2 - (b^2 + c^2 - a^2)^2}$$

This produces one of the more important properties of any triangle, namely that-

$$Area\ Shaded\ Triangle = \frac{1}{4}\sqrt{(2cb)^2 - (b^2 + c^2 - a^2)^2}$$

So, without resorting to any trigonometry and using only the Pythagorean Theorem we are able to express the area of any oblique triangle in terms of the length of its three sides. To test things out consider an equilateral triangle where a=b=c. It says the area will be-

Area Equilateral Triangle= $\frac{\sqrt{3}}{4}a^2$

Also for a=3, b=4, and c=5 which produces a right triangle, we have –

Area $= \frac{1}{4}\sqrt{40^2 - 32^2} = 6$

The above formula for the shaded oblique triangle is equivalent to Heron's Formula which states that-

Area Any Triangle $= \sqrt{s(s-a)(s-b)(s-c)}$

, where s=(a+b+c)/2 is the semi-perimeter. I remember asking my math teacher in high school some 63 years ago how Heron was able to come up with his formula some 2000 years ago. She did not know. I later learned that it involves looking at a circle inscribed in the triangle.

Properties of Oblique Triangles:

Going back to the above figure, one can readily show that at point P the sum of the three angles around it and lying within the rectangle equal 180 degrees. Thus we have the triangle angle rule that-

A+B+C=180 degrees= π radians.

Note that the sum of the interior angles for the enclosing box is 4x90=360deg=2π radians. More generally the sum of the angles of any n sided polygon will equal (n-2)π radians. The sum of the interior angles of a hexagon will be 720deg=4π radians.

To relate the side lengths and the three corner angles to each other one needs to introduce the trigonometric functions. These are associated with right triangles having vertex angles ofA,B and $\pi/2$. The Pythagorean Theorem $a^2+b^2=c^2$ is always valid for right triangles. The longest length c of a right triangle is referred to as its hypotenuse. The three most important of the trigonometric functions area-

$$\sin(\theta)=\frac{opposite\ length}{hypotenuse}\ ,\ \cos(\theta)=\frac{base\ length}{hpotenuse}\ and\ \tan(\theta)=\frac{opposite\ length}{base\ length}$$

From these definitions we have at once from the two right triangles inside the oblique triangle that-

$$\sin(C)=\frac{h}{b}\quad and\quad \sin(B)=\frac{h}{c}$$

Eliminating h from these last two equations and recognizing that the concept of a hypotenuse is not applicable for oblique triangles, one arrives at the following Law of Sines -

$$\frac{\sin(A)}{a}=\frac{\sin(B)}{b}=\frac{\sin(C)}{c}$$

, which is valid for all triangles.

Going back to the figure and applying the Pythagorean Theorem to the two right triangles within the oblique triangle we have-

$$(a-w_2)^2+h^2=c^2\quad ,\quad h^2+w_2^{\ 2}=b^2\quad ,\quad \cos(C)=\frac{w_2}{b}$$

Eliminating h and w_2 from these three equations produces the Law of Cosines-

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

When C=π/2 radians this just reduces to the Pythagorean Theorem. For an equilateral triangle we have cos(π/3)=cos(60deg)=0.5. According to this law one also knows that one of the acute angles of a 3-4-5 right triangle equals θ=arcos(3/5)=53.130 deg. The other acute angle is ψ=arcos{4/5)=36.869deg . As expected , θ+ψ=90deg=π/2 rad . We could also have gotten θ using the law of sines. There θ=arcsin(4/5).

Some less known identities involving oblique triangles are the Mollweide Formulas. Mollweide lived from 1774 to 1825 and was a professor of mathematics at the University of Leipzig. It's the same university where my father received his PhD back in 1933. Mollweide's formulas are straight forward to derive using the Law of Sines. If one looks at the ratios of (a+b)/c and (a-b)/c and expresses the ratios in terms of angles, one gets-

$$\frac{(a+b)}{c} = \frac{[\sin(A)+\sin(B)]}{\sin(C)} \quad \textit{and} \quad \frac{(a-b)}{c} = \frac{[\sin(A)-\sin(B)]}{\sin(C)}$$

These are the two Mollweide Formulas. They can also be rewritten in terms of half angles, but this does not affect their ability to serve as an accuracy check for the triangles associated with geodetic measurements since they contain all side lengths and angles for any oblique triangle. Let us look at an example where the side-lengths are a=5, b=3 and c=7 . By the Law of Cosines this produces –

cos(A)=11/14 , cos(B)=13/14 , and cos(C)=-1/2

from the first Mollweide Formula we then have-

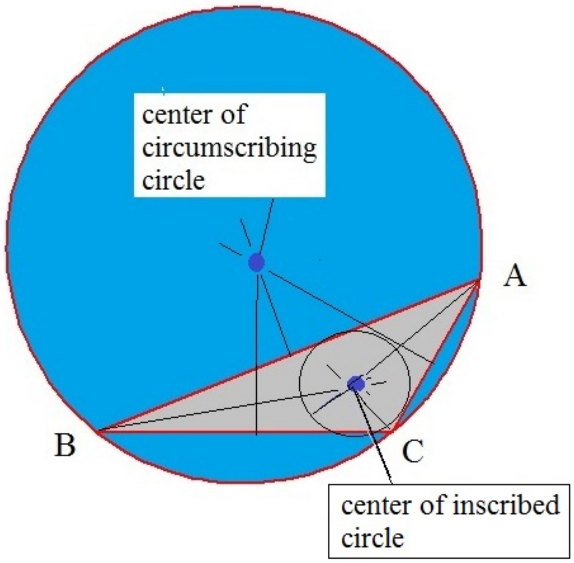
$$\frac{(5+3)}{7} = \frac{\sqrt{14^2 - 11^2} + \sqrt{14^2 - 13^2}}{7\sqrt{3}}$$

which checks out.

Inscribing and Circumscribing Triangles by Circles:

We next ask what is the radius of the smallest circle which can circumscribe an oblique triangle and also what is the radius of the largest circle which can be placed inside the same triangle. We are looking at the following picture-

FINDING THE CIRCLES WHICH INSCRIBE AND CIRCUMSCRIBE AN OBLIQUE TRIANGLE



We notice that the two circles are constructed by two different methods. The larger circle which circumscribes the oblique triangle ABC has its center determined by the intersection of three lines each intersecting the middle of the triangle sides at right angles. The radial distance from this circle center at [x_c,y_c] to the triangle vertexes have a constant value of R. The triangle side ‘a’ is kept horizontal with the coordinates at A, B , and C given by [x3,y3], [0,0], and [[a,0], respectively. After some manipulations one finds-

$$x_c = \frac{a}{2} \quad , \quad y_c = \frac{[x_3(x_3 - a) + {y_3}^2]}{2y_3} \quad \textit{with circle radius} \quad R = y_c$$

To test out this result let us look at an equilateral triangle of side-length a=b=c=1. Here x_3=1/2 and y_3=1/[2sqrt(3)]. This produces the circumscribing circle-

$$[x - (1/2)]^2 + [y - (1/2\sqrt{3})]^2 = (\sqrt{3})^2$$

The circle center lies at the centroid of the equilateral triangle as expected. The distance from the centroid to any of the three corners equals 1/sqrt(3).

We next look at finding the largest circle which can be inscribed in any oblique triangle. We expect this circle to be centered at the intersection of the three lines which bisect the angles A, B, and C. Using the Pythagorean Theorem, the Law of Cosines, and the Law of Sines, one finds after some calculations that the inscribed circle is given by-

$$(x-x_c)^2+(y-y_c)^2=y_c{}^2$$

, with-

$$x_c=\frac{a}{1+\left[\dfrac{\tan(\varphi)}{\tan(\psi)}\right]} \quad and \quad y_c=\frac{a\tan(\varphi)}{1+\left[\dfrac{\tan(\varphi)}{\tan(\psi)}\right]}$$

Here ϕ and ψ are the half angles of B and C , respectively. They are given explicitly as-

$$\varphi=\arccos\left\{\frac{1}{2}\sqrt{\frac{2ac+a^2+c^2-b^2}{ac}}\right\} \quad and \quad \psi=\arccos\left\{\frac{1}{2}\sqrt{\frac{2ab+a^2+b^2-c^2}{ab}}\right\}$$

For the special case of an equilateral triangle with a=b=c=1 we get x_c=1/2 and y_c=1/[2sqrt(3)]. That is, the circle centers at [1/2, 1/2sqrt(3)] just like for the circumscribed circle. However this time the radius has the smaller value of-

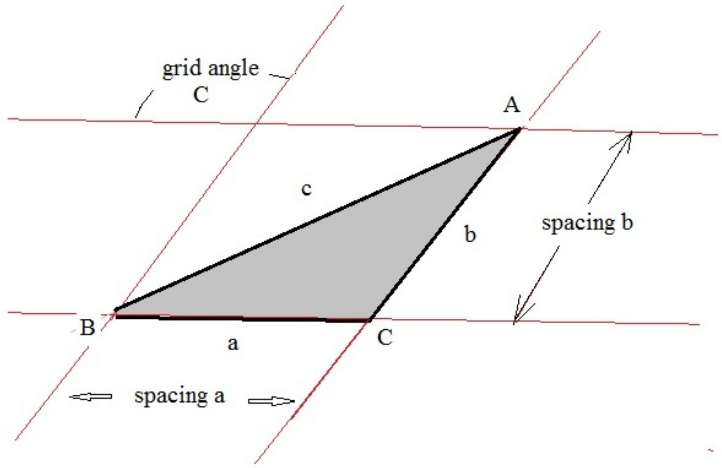
$$R=y_c=\frac{1}{2\sqrt{3}}$$

In general the circumscribing and inscribing circle of an oblique triangle will not coincide.

Tiling with Oblique Triangles:

The simplest form of tiling which leaves no gaps is accomplished by using rectangular or square plates. What is less often recognized is that any tile having the shape of an oblique triangle can also be used to tile a flat floor without leaving gaps. Let us demonstrate how this is done. One begins with the oblique triangle shown in the first figure above and draws two straight lines containing sides a and b to ±infinity as shown-

TILING WITH OBLIQUE TRIANGLES



Area of Rhomboid=[a-bcos(C)]bsin(C)

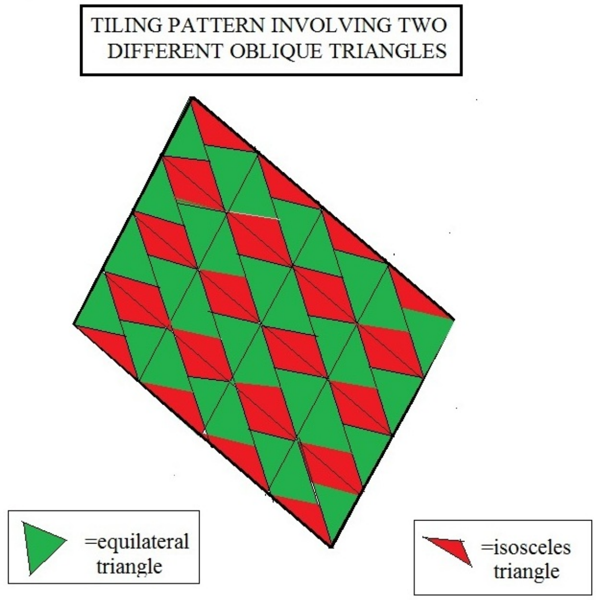
Next equally spaced parallel lines are drawn to these to form a Rhomboidal grid. Each rhomboidal element contains two identical oblique triangles formed by drawing a diagonal line from A to B. The area of each rhomboidal element is equal to-

$$\text{Area Rhomboid=absinC} = \frac{1}{2}\sqrt{(2ab)^2 - (a^2 + b^2 - c^2)^2}$$

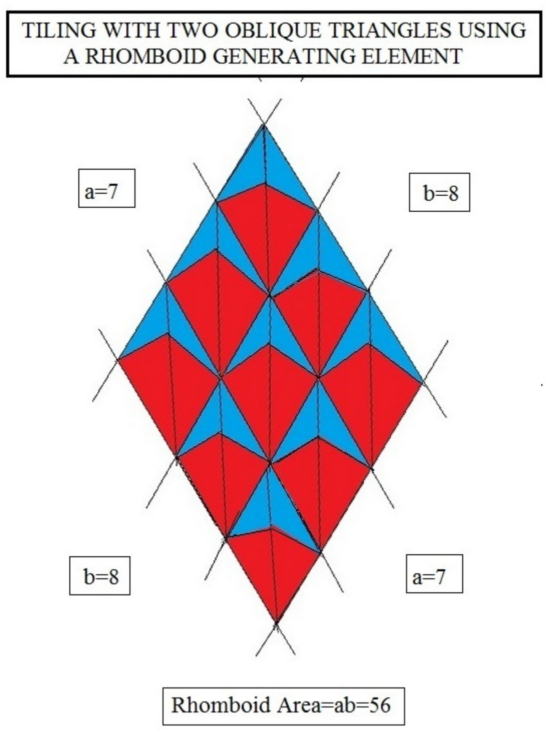
, so that the area of each identical oblique triangle will be just half of this.

Although this result looks slightly different from the area formula we derived earlier in this article, the results yield the same value. An oblique triangle with a=5, b=3, and c=7 yields an area of (15/4)sqrt(3)=6.495… .

One could also break up the existing triangle ABC into two sub-triangles to produce a gapless tiling with two different oblique triangle tiles. Alternatively one could generate two different oblique triangles by cutting a 1–sqrt(3)-2 right triangle with a single line emanating at C. A resultant two tile pattern follows-



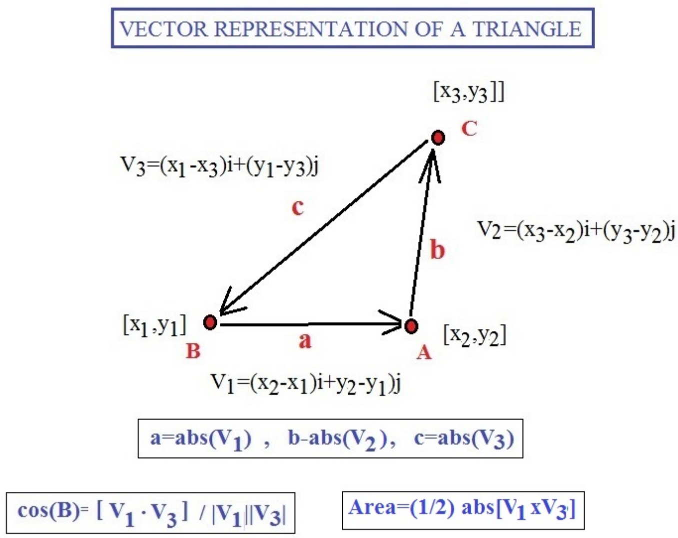
Here the tiles have the shape of an isosceles triangle and an equilateral triangle. Another type of tiling reminiscent of Penrose Tiling but differing from it by having a perfectly periodic pattern is the following-



Here the kite and arrow sub-elements each contain two identical oblique triangles.

Using Vectors to Define Triangles:

A convenient way to handle many problems involving triangles is to employ vector forms for its sides. One can define three vectors as shown in the following figure-



Here the absolute values of the vectors represent the side-length of the triangle so that a=abs(V_1)=$|V_1|$ etc. .. The cosine of a particular angle is obtainable by using the dot product-

$$\cos(B) = [V_1 \cdot V_3]/[|V_1||V_3|]$$

For an [a,b,c]=[5,3,7] triangle, we have-

$$49 = (x_3)^2 + (y_3)^2 \quad and \quad 9 = (x_3 - 5)^2 + (y_3)^2$$

This means [x_3,y_3]=[13/2, sqrt(29)/2]. So we get-

$$\cos(B) = \{(5i) \cdot [(13/2)i + (\sqrt{29}/2)j]\}/35 = 13/14$$

or B=21.7867 deg. We can verify this result by using the Law of Cosines. There we have-

$$\cos(B) = \frac{(25 + 49 - 9)}{2(35)} = 13/14$$

Taking the absolute value of the cross product of vectors V_1 and V_3 one finds the area of a rhomboid with sides $|V_1|$ and $|V_3|$. The area of the triangle formed is just half of this value. For the above case we get-

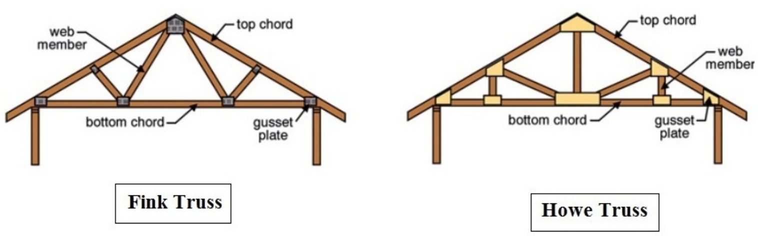
$$\text{Area Triangle}=\frac{1}{2}abs\begin{vmatrix} i & j & k \\ 5 & 0 & 0 \\ 13/2 & \sqrt{29}/2 & 0 \end{vmatrix}=\frac{5\sqrt{29}}{4}=6.7314..$$

For an equilateral triangle of side length a=b=c=1 we find its area to be sqrt(3)/4.

Some Practical Applications of Triangles:

Besides the mathematical aspects of triangles discussed above, triangles play important practical roles in connection with trusses and tripods. Trusses are used in railway bridges, towers, and buildings. A typical truss is composed of interlocking triangles as the following picture of typical roof trusses shows-

EXAMPLES OF TWO TYPES OF ROOF TRUSSES



Note the basic triangular shapes of the sub-sections

(source-http://www.ashireporter.org)

A typical truss consists of triangular shaped sub-elements which will not distort when subjected to loads or moments. This is unlike what would happen to a rectangular sub-sections when subjected to a rotational torque without external constraints. Triangular truss components carry only compressive loads along their beams and hence do not require very strong gusset plates to hold its ends together. Some trusses have members which carry no load such as the two short pieces in the above Howe truss shown. Such zero force pieces have no functional purpose other than to keep the truss from warping out of its plane.

The tripod is another practical device involving a triangle. The essence of a tripod are three legs which come together at a point where a camera or one's eye level is located. The other ends of the legs form an oblique triangle at ground level. Here is a drawing of an early land surveyor (such as George Washington in his late teens)-

COLONIAL LAND SURVEYER USING A LARGE TRIPOD

Note how the three legs of the tripod define an oblique triangle at ground level

The unique property of a tripod is that location of the three leg bottoms are sufficient to define a unique plane in space no matter how rough the ground . This observation stems from the fact that it takes only three points to define any plane.. Extra legs are not needed to define a plane and may even detract for finding a single plane to, for example, keep a four legged table from rocking. Many of you have probably had the experience when dining at an outdoor cafe of having your drink spill because you happen to be sitting at a table where one of the legs is not touching the uneven ground. Under such a condition one is dealing with different ground planes defined by just three legs at a time. My typical solution is to fold a napkin several times and then place it under the shortest leg relative to a plane defined by the three other legs. Someone should invent a little mechanical ratchet attachment capable of lengthening or shorting until all four leg bottoms find themselves in the same plane no matter how rough the ground. An alternative way of having a stable table would be to have just a single leg attached to a heavy round disc base and attach to the bottom of this disc three very short support points. In my wood shop I often find myself building bookshelves, chairs and tables with four legs not precisely of the same length. To solve this problem I turn the device upside down and then use a level lying on a rigid yard stick to mark the legs at the same height above the ground. Very often one can use self-sticking felt pads to adjust the height difference without needing to saw anything.

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