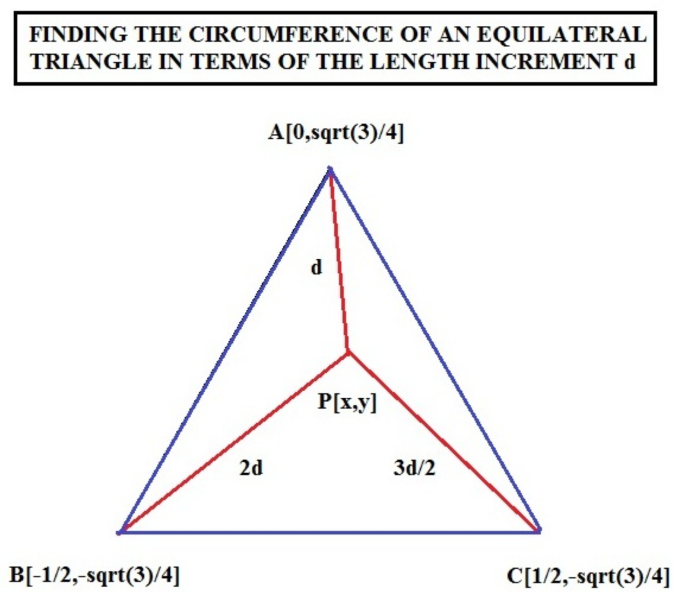
RELATING LINE LENGTH WITHIN AN EQUILATERAL TRIANGLE TO ITS CIRCUMFERENCE

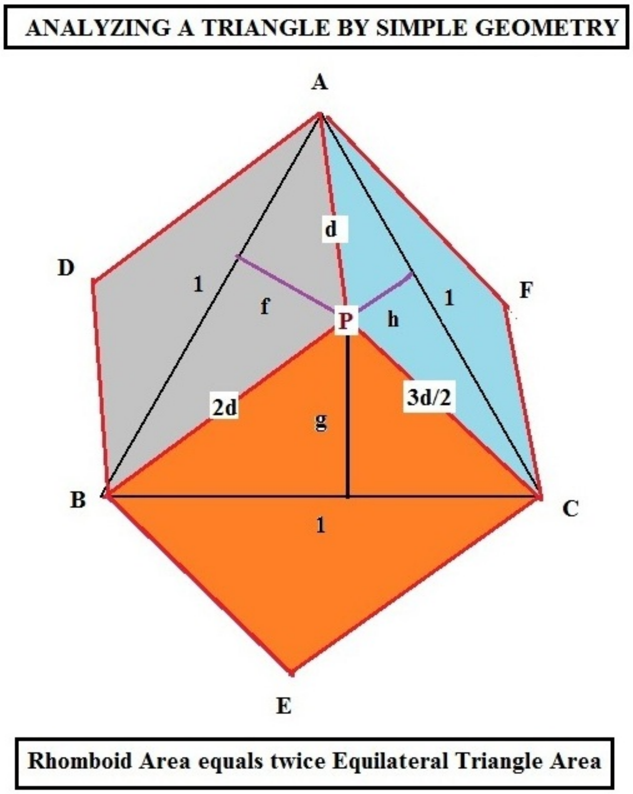
The Wall Street Journal usually publishes a mathematical puzzle page in its Saturday edition. Sometimes these problems are quite challenging although they require only some elementary knowledge of algebra and geometry. Here is one of these problems appearing in the May 28, 2016 issue of the Journal. I'll work out the answer for you using several different approaches.

The problem deals with an equilateral triangle whose sides can be chosen without loss of generality as having length one each. As shown in the following figure, the triangle is cut by lines emanating from each of its three vertexes and meeting at a common point P[x,y] inside the triangle-



The lines shown in red have lengths d, 2d and 3d/2. One wants to find d and then relate it to the total triangle circumference of three. A second problem is to determine the sum of the length increments drawn from interior point P perpendicular to the three edges.

There are several ways one may attack this problem. The simplest is via elementary geometry based on the use of rhomboids. This is the approach one would expect a typical Wall Street Journal reader to use. The starting figure is the following-



By looking at this figure it is obvious that the equilateral triangle area ABC equals sqrt(3)/8 when the triangle has sides of length one each. Furthermore each rhombus has an area twice that of the sub-triangles ABP, BCP, and CAP. Also the area of all three sub-triangles is given as-

$$(\tfrac{1}{2})\cdot(f+g+h)=\mathrm{Area}_{ABC}=\mathrm{sqrt}(3)/8$$

So, without any elaborate math, we have shown that the sum of f+g+h=sqrt(3)/4. This answers part of the puzzle. Next consider finding d by a similar geometric approach. Looking at the three sub- triangles one finds, with application of the Pythagorean Theorem, that-

$$f=\frac{1}{2}\sqrt{10d^2-9d^4-1} \qquad g=\frac{1}{8}\sqrt{200d^2--49d^4-16}$$

$$and \quad h=\frac{1}{8}\sqrt{104d^2-25d^4-16}$$

Thus we arrive at the rather complicated result-

$$2\sqrt{3}=$$
$$4\sqrt{10d^2-9d^4-1}+\sqrt{200d^2-49d^4-16}+\sqrt{104d^2-25d^4-16}$$

Solving this by computer we find $d=0.4035482122$. Note from the figure geometry d must be larger than (2/5)=0.400 and smaller than sqrt(3)/4=0.433.

Thus the puzzle has been solved in closed form using only a little geometry and the theorem that a2+b^2=c^2 for any right triangle.

The above simple approach to the problem solution is not the only one which can be used. We could for instance draw three circles centered on the triangle vertexes having radii d, 2d, and 3d/2. Analytically this produces the equations-

$$(x-0)^2+(y-\frac{\sqrt{3}}{4})^2=d^2$$
$$(x+\frac{1}{2})^2+(y+\frac{\sqrt{3}}{4})^2=4d^2$$
$$(x-\frac{1}{2})^2+(y+\frac{\sqrt{3}}{4})^2=\frac{9}{4}d^2$$

By first eliminating the y term, one finds x=(7/8)d^2. Next solving for y, we have y=(17d^{2-}2)/(8sqrt(3)). So from the first equation we get-

$$109d^4-116d^2+16=0$$

This algebraic equation has four real roots with the one of interest being-

$$d=0.4035482122$$

Thus the triangle circumference of 3 equals 7.434ds.

A still other way to solve this problem involves adding the areas of the three sub-triangles lying within the equilateral triangle and setting the sum to sqrt(3)/8.The sub-triangle areas can be gotten by the Heron‘s Formula which says that the area of any triangle of side-lengths a, b, and c is given by-

$$TriangleArea=\sqrt{s(s-a)(s-b)(s-c)}\quad with\quad s=(a+b+c)/2$$

For the present case we have-

$$2\sqrt{3} =$$

$$4\sqrt{10d^2 - 9d^4 - 1} + \sqrt{200d^2 - 49d^4 - 16} + \sqrt{104d^2 - 25d^4 - 16}$$

Which is the same form we found earlier. The solution remains d=0.4035482122…

By redefining the three line lengths from d, 2d, and 3d/2 to d, d, and d, one places the point P[x,y] at the center of the equilateral triangle at [x,y]=[0,-1/(4sqrt(3))]. The smallest circle circumscribing this triangle will have radius 1/sqrt(3) and the largest radius circle inscribable within the triangle will equal to 1/(2sqrt(3)).

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