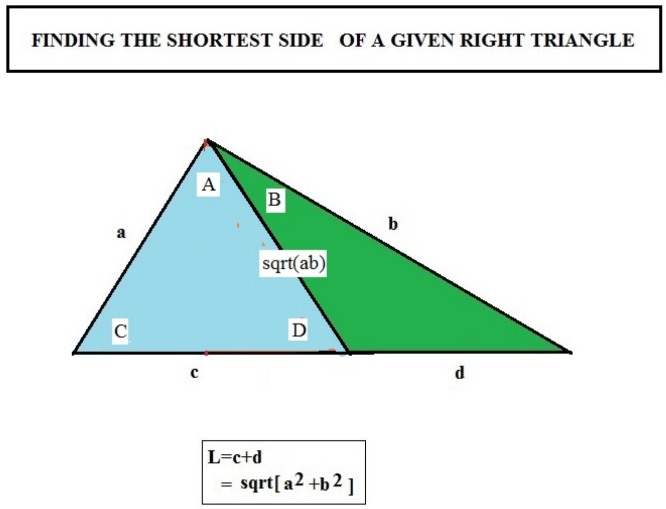
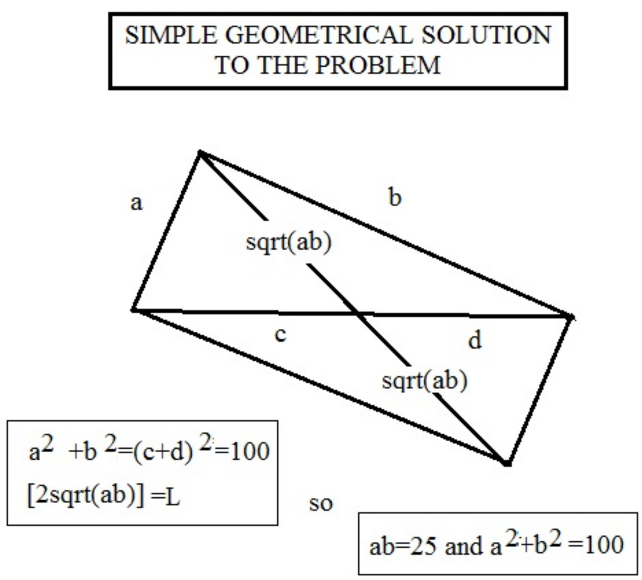
DETERMINING THE SIDE LENGTH OF A RIGHT TRIANGLE

In yesterday's Wall Street Journal Puzzle Page (Sept 10-11, 2016) the question was asked to find the length of the shortest side 'a' of a right triangle a-b-L, where the hypotenuse length L=10 and a median drawn from the right angle to the middle of the hypotenuse has length sqrt(ab). A schematic of the problem follows-



The simplest way to solve this problem is by use of problem symmetry and some geometry. We first draw the following picture-



Here we have considered the large right triangle to be half a box of area ab. The diagonals are equal so that c+d=10= 2sqrt(ab). So we find ab=25. Also from the Pythagorean Theorem we have $a^2+b^2=100$. Solving for a and b we get-

$$a = 5\sqrt{2-\sqrt{3}}.. \qquad and \qquad b = 5\sqrt{2+\sqrt{3}}$$

Note that a<b.

The second approach to solving the problem is to use the full repertoire of trigonometric formulas. It starts with the fact that-

$$\sin(C) = \frac{b}{L} \qquad and \qquad \cos(C) = \frac{a}{L} \qquad with \qquad L = \sqrt{a^2+b^2} = c+d$$

Next, using the Law of Sines for the light blue triangle produces-

$$\frac{\sin(A)}{c} = \frac{\sin(D)}{a} = \frac{\sin(C)}{\sqrt{ab}}$$

This simplifies to -

$$\sin(D) = \frac{\sqrt{ab}}{L} \qquad and \qquad \sin(A) = \frac{c}{L}\sqrt{\frac{b}{a}}$$

For the green sub-triangle we find-

$$\frac{\sin(\pi/2 - C)}{\sqrt{ab}} = \frac{\sin(\pi - D)}{b} = \frac{\sin(\pi/2 - A)}{d}$$

We can simplify this last result to-

$$\frac{\cos(C)}{\sqrt{ab}} = \frac{\sin(D)}{b} = \frac{\cos(A)}{d}$$

Combining things, this produces-

$$\cos(A) = \frac{d}{L}\sqrt{\frac{a}{b}}$$

Next applying the Law of Cosines to the light blue triangle we get-

$$c^2 = a^2 + ab - 2a\sqrt{ab}\cos(A) = ab + a^2(1 - 2d/L)$$

This yields a quadratic in c which reads-

$$c^2 - (\frac{2a^2}{L})c + a(a-b) = 0$$

But the puzzle states that c=d=L/2=5. This produces-

$$ab = 25 \qquad withd \qquad a^2 + b^2 = 100$$

Solving for a and b we get the same answer as earlier, namely,-

$$a = 5\sqrt{2-\sqrt{3}} = 2.588.. \qquad and \qquad b = 5\sqrt{2+\sqrt{3}} = 9.659... \quad .$$

Thus the shortest leg equals a=2.588 . Also we have sqrt(ab)=5=L/2, so that both the light blue and green sub-triangles are isosceles triangles of different height.

We can call the two solution approaches above as the Thomas Edison approach to that of one of his highly trained imported European mathematicians. The story goes that Edison wanted to know the volume of one of his light bulbs and put an imported assistant to work on the problem. After a week of heavy calculations the assistant came back very proudly announcing the right answer after hours of elaborate calculations using a model based on the concatenation of ellipsoids with cylinders. Edison’s reply was “Why didn’t you just fill the bulb with water and then measure the water volume with a graduated beaker?”.

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