One of the more important trigonometric functions is arctan(x) defined by-

\[ \arctan(x) = \int_{t=0}^{x} \frac{dt}{1+t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)} \]

Using the double angle formula for tangent one also has that-

\[ \arctan(x) + \arctan(y) = \arctan\left(\frac{x + y}{1 - xy}\right) \]

From this equality follow the two identities-

\[ \arctan(x) = \frac{1}{2} \arctan\left(\frac{2x}{1 - x^2}\right) \]

and-

\[ \arctan(x) + \arctan\left(\frac{1}{x}\right) = \arctan(\infty) = \frac{\pi}{2} \]

A plot of \( \arctan(x) \) in \( 0<x<6 \) looks like this-
From the graph and from the above series one sees that arctan(x) is an odd function with \( \text{arctan}(x) = -\text{arctan}(-x) \) and it has values ranging from \( y(-\infty) = -\frac{\pi}{2} \) through \( y(0) = 0 \) to \( y(\infty) = \frac{\pi}{2} \). One also sees that accurate values of arctan(x) need only be found in the range \( 0 < x < 1 \) since the values for the remaining ranges of x will be known from the above formulas. So, for example, if we know the value of arctan(1/2) one has the values of arctan(2), arctan(-1/2), arctan(-2), arctan(4/3), arctan(-4/3), etc.

There are certain points within \( 1 \geq x \geq 0 \) for which arctan(x) is known exactly without needing to go to a numerical approximation. These include:

\[
\begin{align*}
\text{arctan}(1) &= \frac{\pi}{4} = 0.78539816... \\
\text{arctan}\left(\frac{1}{\sqrt{3}}\right) &= \frac{\pi}{6} = 0.52359877... \\
\text{arctan}(\sqrt{2} - 1) &= \frac{\pi}{8} = 0.39269908... \\
\text{arctan}(2 - \sqrt{3}) &= \frac{\pi}{12} = 0.26179938...
\end{align*}
\]

Further exact values may be generated by the formula-
\[
\frac{1}{2} \arctan(x) = \arctan \left( \frac{-1 + \sqrt{1 + x^2}}{x} \right)
\]

or by combining some of the above terms. For example, we find the single term formula for \(\pi\) -

\[
\pi = 24 \arctan \left( \frac{\sqrt{6} - \sqrt{2} - 1}{2 - \sqrt{3}} \right)
\]

To evaluate \(\arctan(x)\) for values of \(x\) for which no convenient exact value exists but which lies close to a known value \(\arctan(x_0)\), one can make use of the identity-

\[
\arctan(x) = \arctan(x_0) + \arctan \left( \frac{\varepsilon}{1 + x x_0} \right)
\]

where \(\varepsilon = x - x_0 \ll x_0\). For abbreviation sake, we define the new parameter-

\[
b = \frac{\varepsilon}{1 + x x_0} = \frac{\varepsilon}{x_0^2 + \varepsilon x_0 + 1} \ll 1
\]

and now concentrate on how to find \(\arctan(b)\). If \(b\) is extremely small then the first few terms of the standard \(\arctan\) series will yield excellent results. However for larger but still small values of \(b\) such a series approximation becomes impractical when accuracies of the order of twenty digits or higher are required. In the latter case we resort to a new approach introduced by us several years ago. The method is based upon the use of even Legendre polynomials appearing in the integral-

\[
I(n, b) = \int_{\varepsilon=0}^{1} \frac{P_{2n}(t)}{\varepsilon^2 + \varepsilon t^2} \, dt = M(n, b) - N(n, b) \arctan(b)
\]

The even Legendre polynomials are oscillatory functions with \(n\) zeroes in \(0 < t < 1\). When \(b\) gets small the dominant term in the denominator is \(1/b^2\) and the integral will approach a value of zero since-

\[
J(n) = \text{const.} \int_{\varepsilon=0}^{1} P_{2n}(t) \, dt = 0 \quad \text{for} \quad n = 1, 2, 3, 4, \ldots
\]

Note that this would not be the case if we dealt with odd Legendre polynomials.
Setting the integral \( I(n,b) \) to zero, leads to the arctan approximation:

\[
\arctan(b) \approx \frac{M(n,b)}{N(n,b)} = R(n,b)
\]

The integral can be evaluated analytically to yield the ratios:

\[
R(1,b) = \frac{3b}{b^2 + 3}
\]

\[
R(2,b) = \frac{(55b^3 + 105b)}{(9b^4 + 90b^2 + 105)}
\]

\[
R(3,b) = \frac{(231b^5 + 1190b^3 + 1155b)}{(25b^6 + 525b^4 + 1575b^2 + 1155)}
\]

and

\[
R(4,b) = \frac{(15159b^7 + 147455b^5 + 345345b^3 + 225225b)}{(1225b^8 + 44100b^6 + 242550b^4 + 420420b^2 + 225225)}
\]

These ratios yield continuously improving approximations for \( \arctan(b) \) as \( n \) gets large and \( b \) remains small. The quotients are somewhat reminiscent of Pade approximates but are much easier to derive than the latter for the same order of accuracy. You will note that as \( b \) approaches zero all values of \( R(n,b) \) go as \( b \). Also one has \( \arctan(x) = \arctan(x_0) + R(\infty,b) \).

To demonstrate the accuracy of these approximations consider the case of finding the value of \( \arctan(0.4) \) to more than 20 places. Here we have –

\[
\arctan(0.4) = \arctan(\sqrt{2} - 1) + \arctan(b) \quad \text{with} \quad b = \frac{1.4 - \sqrt{2}}{1 + 0.4(\sqrt{2} - 1)} = -\frac{1}{41 + 29\sqrt{2}}
\]

This yields the approximation:

\[
\arctan(0.4) \approx \frac{\pi}{8} + R(3, \frac{-1}{41 + 29\sqrt{2}}) = 0.3805063771123648863035879168590118605246
\]

Comparing this to the 40 place computer evaluation –

\[
\arctan(0.4) = 0.3805063771123648863035879168104331044974
\]

shows that the \( R(3,b) \) approximation already yields a 28 digit accurate result. Going to \( R(4,b) \) yields the even better approximation:

\[
\arctan(0.4) \approx 0.3805063771123648863035879168104331045651
\]
which is accurate to 36 places. Further improvements are possible by going to still larger \( n \). however, the quotients \( R(n,b) \) become progressively larger making an evaluation more difficult.

There are numerous arctan formulas available for evaluating \( \pi \). Among the earliest and simplest is the Leibnitz Formula-

\[
\pi = 4[\arctan(\frac{1}{2}) + \arctan(\frac{1}{3})]
\]

Although the series for these two arctan terms converge slowly, one can speed things up by expanding about the exact values \( \arctan(1/\sqrt{3}) \) and \( \arctan(\sqrt{2}-1) \). This produces the equivalent form-

\[
\pi = 24[\arctan(\frac{1}{8+5\sqrt{3}}) + \arctan(\frac{1}{7+5\sqrt{2}})]
\]

whose series converge much more rapidly. Using our \( R(n,b) \) approximation we can estimate the value of \( \pi \) to be-

\[
\pi \approx 24[R(n, \frac{1}{8+5\sqrt{5}}) + R(n, \frac{1}{7+5\sqrt{2}})]
\]

It yields the eighteen digit accurate result-

\[
\pi = 3.14159265358979323
\]

when \( n \) is taken as 3. Better results could be gotten by choosing \( n \) larger or evaluating things when \( b_1 \) and \( b_2 \) are made smaller. They had values of 1/19.180 and 1/14.071 in this calculation.

Finally we look at finding the value of \( \pi \) using the even simpler Gregory formula-

\[
\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots
\]

This infinite series converges but does so at an extremely slow pace making it useless for finding \( \pi \). However one can first do some telescoping of terms to bring things near an exact known value and then apply the Legendre integral approach. We find-

\[
\frac{\pi}{4} = \arctan(1) = 2 \arctan(\sqrt{2} - 1) = 4 \arctan(\frac{-1 + \sqrt{4 - 2\sqrt{2}}}{\sqrt{2} - 1})
\]
The term in the last square bracket equals about K=0.19634 suggesting that we use x₀ = 0.2. Thus we have the approximation for Pi (after modifying the Gregory Formula), of-

\[ \pi \approx 16[\arctan(0.2) + R(n, \frac{K - 0.2}{1 + 0.2K})] \]

The value of \( \arctan(0.2) \) needs to still be found using a similar operation to that used earlier for finding \( \arctan(0.4) \). It is also interesting to note that the same ratio R combined with the Machin Formula \( \pi = 16\arctan(1/5) - 4\arctan(1/239) \) produces the additional approximation-

\[ \arctan\left(\frac{1}{239}\right) \approx -4R(3, \frac{K - 0.2}{1 + 0.2K}) = 0.0041840760020747238645382149 \]

April 2014