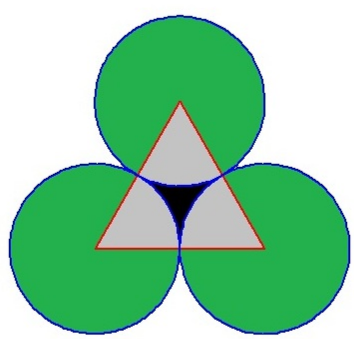
VOID FRACTIONS CREATED BY THE BUNDLING Of REGULAR POLYGONS

It is well known equilateral triangles, squares, and regular hexagons can be arranged so that a collection of each of these can fit nicely into a plane without leaving any gaps. This will no longer be the case for regular pentagons, heptagons, octagons, and certainly not for circles which can be thought of as a regular polygons with an infinite number of sides.

There are numerous problems were one needs to know the area of the spaces (voids) left over when n-sided regular polygons are bundled together. This occurs for instance when cooling super-conducting cables or in thermal pumping where oscillating fluid is confined to capillary bundles of circular cross-section. It is our purpose here to study this problem of voids created by bundled regular polygons.

Let us begin by examining the cross-section of a collection of closely packed capillary bundles or a bundle of copper wires. In both cases the cross-section base structure consists of three circles of radius r arranged in a ring as shown-

VOID LEFT BY CLOSELY-PACKED CIRCLES AS DETERMINED FROM ITS TRIANGULAR BASE



void fraction $\varepsilon = 1 - \pi/[2\mathrm{sqrt}(3)] = 0.09310$

The black scalloped triangle represents the void in this instance. You can get an additional feel for this configuration by taking three pennies and arranging them as indicated. If the circles are taken to represent super conducting wires then the voids can be taken as the conduits for carrying liquid helium to keep the wires superconducting. For thermal pumping were liquids such as water are oscillated axially one typically blocks off the void areas.

Calculating the void fraction in the above configuration and others, we start off with the general formula for any regular polygon of n sides of side-length s each. The area of any of these polygons is-

$$A_n = \frac{ns^2}{4}\cot(\frac{\pi}{n})$$

Now a circle can be thought of as an infinite sided regular polygon. Thus the above formula yields-

$$A_\infty = \frac{s^2}{4}\frac{\lim}{n \to \infty}\left[\frac{n}{\tan\left(\frac{\pi}{n}\right)}\right] = \pi\, r^2$$

, since ns represents a circle’s circumference in the limit. To find the void area we note that the triangle has area sqrt(3) r^2 and the grey area within the triangle equals $\pi r^2/2$. Thus the void area is-

$$A_{void} = \sqrt{3}r^2 - \frac{\pi\, r^2}{2} = r^2(\sqrt{3} - \frac{\pi}{2})$$

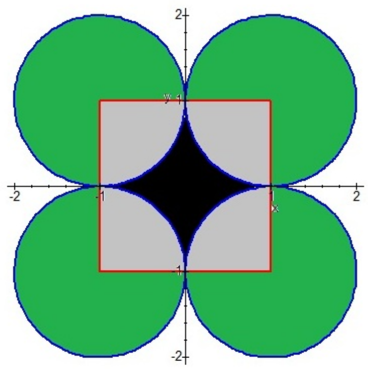
and the non-dimensional void fraction becomes-

$$\varepsilon = \frac{A_{void}}{A_{triangle}} = 1 - \frac{\pi\sqrt{3}}{6} = 0.0931...$$

Thus a little less than 10% of the cross section of closely packed circles is left over for voids.

We next go to the only other possible periodic array of packed circles. It is the square based configuration shown-

VOID LEFT OVER WHEN DEALING WITH A SQUARE LATTICE OF CIRCLES



void fraction $\varepsilon = 1 - (\pi/4) = 0.21460..$

Here the square base has area $4r^2$ and the grey area yields an area of $\pi r^{2.}$ Thus the void faction becomes-

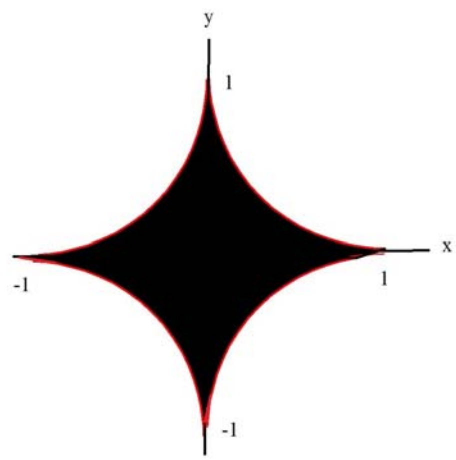
$$\varepsilon = \frac{4-\pi}{4} = 1 - \frac{\pi}{4} = 0.21460...$$

This value is noted to be about twice the void fraction found under close-packed conditions. The void at first glance seems to have the shape of the well known Astroid curve. A close inspection however shows the present void has deeper scallops. It is defined by four appropriately placed circles or approximated by the cusp equation-

$$x^p + y^p = 1 \quad with \quad p = 3.5431$$

Here is a closer view of the void-

FOUR CUSP VOID CREATED BY FOUR BUNDLED CIRCLES



$x^p + y^p = 1$ with $p=3.5431$

We next consider the void fractions associated with regular polygons with a finite number of sides. From group theory and the basics of crystallography we can see at once that stacked arrays composed of odd-numbered sided polygons can never produce a periodic array with the exception of n=3 were there are no voids. So although voids will exist for pentagons, heptagons, etc, the void pattern will not be uniform or periodic and hence will be ignored here. As far as even n patterns are concerned, they will be periodic for face centered and for close-packed configurations only as long as continued division by 4 or 6 reduces to a value of 4 (closed squares without void) or 6 (closed hexagons without gaps). Here are the possible cases-

Side number for square centered arrangements:

$4{\cdot}2^n$ - 64-32-16-8-4 will produce periodic voids

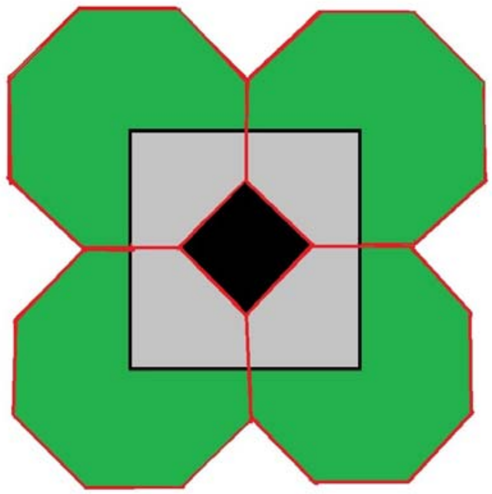
Side number for closed packed arrangements:

$6{\cdot}2^n$-192-96-48-24-12-6 will produce periodic voids

Any other even number polygon will fail to produce a periodic void pattern if repeated divisions by two produces an odd number, or a non-integer. Thus 60-30-15-7.5 and 44-22-11 will not have periodic void patterns.

Consider next a square packed arrangement of identical solid rods of octagonal cross section. This time the void fraction may be calculated with aid of the following figure-

VOID FORMED BY FOUR OCTAGONS



ε =3-2sqrt(2) =0.17157...

The basic building block here is a square containing 4 quarter octagons plus one rotated black square. The area of the containing large square is-

$$A_{square} = s^2\left[1+2\cot(\pi/8)\right] = s^2\left[3+2\sqrt{2}\right]$$

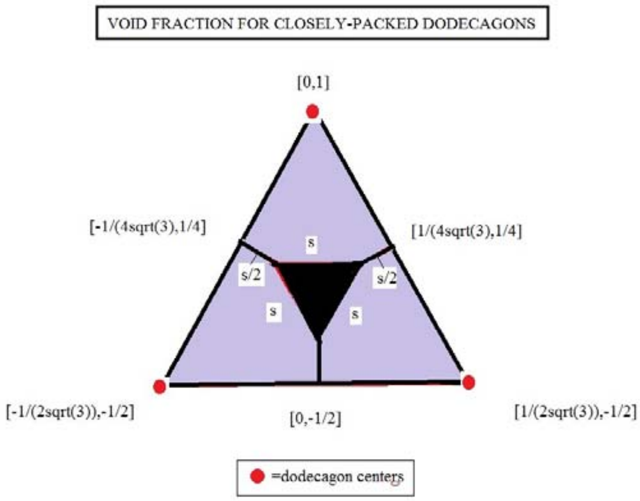
So the void fraction becomes-

$$\varepsilon_8 = \frac{1}{(3+2\sqrt{2}} = \left[3-2\sqrt{2}\right] = 0.17157..$$

since the area of the small square is just s^2 . This epsilon is a little less than the value for square centered circles where ε=0.21460. We can generalize this type of square centered $4{\cdot}2^n$ sided polygon configuration to-

$$\varepsilon_{4\cdot 2^n} = 1 - \frac{2^n}{\cot[\pi/(4\cdot 2^n)]} \qquad n = 1, 2, 3,..$$

Consider next a close-packed array of 12 sided polygons. The base configuration is here an equilateral triangle as shown-



Its side-length equals s cot(π/12) so that the area becomes–

$$A_{triangle} = \frac{\sqrt{3}}{4}\cot(\frac{\pi}{12})^2$$

The grey area within the triangle equals-

$$A_{grey} = \frac{3}{2}s^2\cot(\frac{\pi}{12}) = \frac{3}{2}(2+\sqrt{3})\,s^2$$

The void becomes $A_{triangle}$-A_{grey} , so that the void fraction becomes-

$$\varepsilon_{12} = 1 - \frac{2\sqrt{3}}{2+\sqrt{3}} = 0.071796..$$

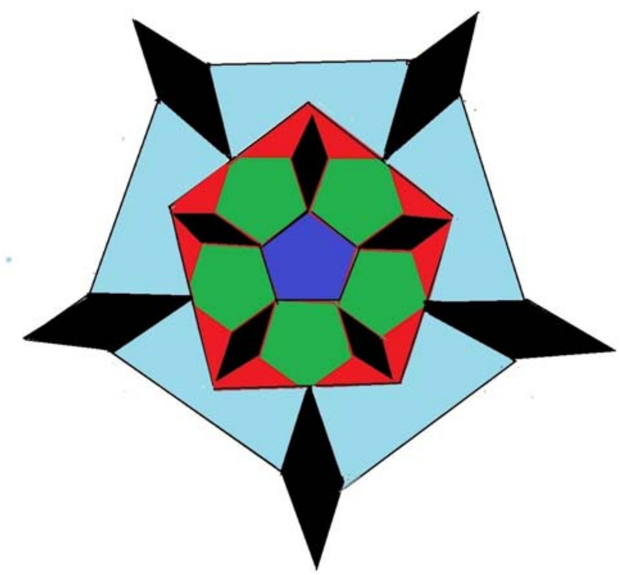
This is a little less than the closely packed circle case where ε=0.093. One can generalize this last result to all bundled close-packed polygons with $6{\cdot}2^n$ sides. The resultant void fraction becomes-

$$\varepsilon_{6\cdot 2^n} = 1 - \frac{\sqrt{3}\,2^n}{\cot(\frac{\pi}{6\cdot 2^n})} \qquad for \quad n = 1,2,3,...$$

Taking the limit as n goes to infinity one recovers the circle results.

We have examined the void fraction for both square centered and close-packed polygon configurations. Those polygon bundles involving polygons whose side number reduce to 4 or 6 upon repeated division by two, produce voids periodic in the x and y directions. Their void fractions increase continually with increasing n reaching their maximum values as n approaches infinity. Those polygons with an odd number of sides do not lead to periodic void arrays although void patterns do exist. They can produce some interesting figures such as the following pentagon configuration-

NESTED PENTAGONS SHOWING NON-PERIODIC VOID PATTERN



This last figure can also be thought of as non-symmetric tiling in the plane. There will be no gaps but the tile elements will occur in a non-periodic manner and will be continually growing in size as the tiling expands outward.

U.H.Kurzweg
Sept. 21, 2017
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