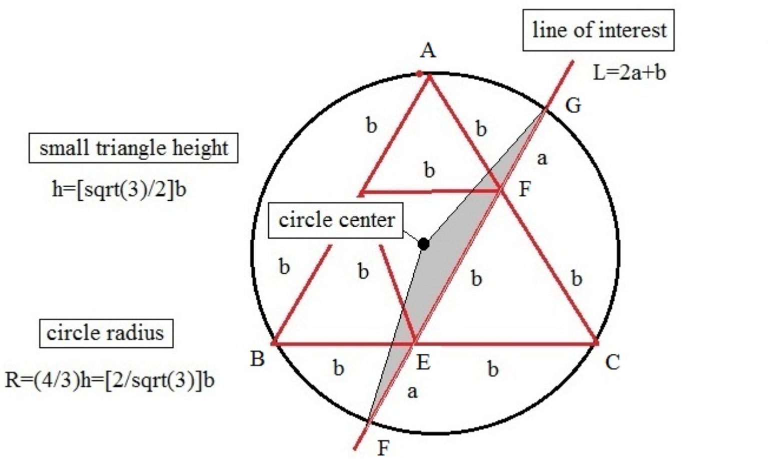
SOLUTION OF A MATH PROBLEM FROM THE AUGUST 16TH ISSUE OF THE WALL STREET JOURNAL

A problem appearing in yesterday's Wall Street Journal in their weekend puzzle page deals with finding the length increments of a line inside a circle with inscribed equilateral triangle. The problem is easiest to describe via the following diagram-

SCHEMATIC FOR FINDING LENGTH b GIVEN THE VALUE OF a



(from Wall Street Journal Puzzle Page, August 13, 2016)

We show here a slightly modified schematic of the problem where we have added four smaller equal sized equilateral triangles of sides b each. Note that the height of the small triangles equals h=b cos(30)=[sqrt(3)/2]b and the circle radius is R=[4/3]h. The question asked is "What is the value of a and b for the line FG" ?. The simplest way to solve this problem is to note that the line length FG equals-

$$L=2a+b$$

and that the perpendicular distance from the circle center to line FG is –

$$\delta=(1/3)h=\{1/[2\mathrm{sqrt}(3)]\}b$$

Next breaking the grey shaded oblique triangle into two equal halves we get from the Pythagorean Theorem that-

$$R^2=\delta^2+(\frac{L}{2})^2$$

Plugging in for R, δ, and L, this produces the identity-

$$5b^2 = (2a+b)^2$$

which is equivalent to –

$$b = \left[\frac{1+\sqrt{5}}{2}\right]a$$

Those of you a bit more familiar with basic math will recognize the term in the square bracket as just the Golden Ratio-

φ=1.61803398…

In the WSJ Puzle page the value of a is specified as a=2 so that their sought after answer is –

b=2φ=3.23606797…

There are other ways to generate this answer. All that is required to keep in mind is that L=2a+b and $R^2=(b/3)^2+(L/2)^2$. The relation between R and b can then be gotten by several routes. Looking at the oblique triangle A-circle center-B and applying the Law of Cosines we find-

$$4b^2 = 2R^2[1+\frac{1}{2}] = 3R^2$$

Substituting into the equations for L and R then produces the identical result that b=φa.

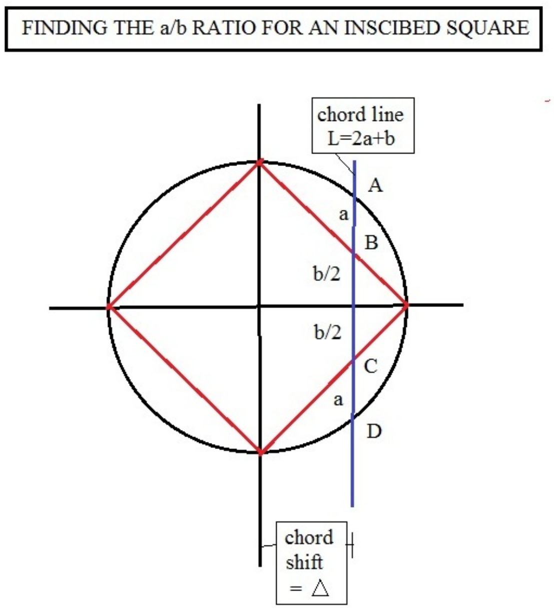
The chords cut by the large triangle have the lengths-

AB=AC=BC=2b=sqrt(3)R

The area of the segment between a chord and the circle is just equal to one-third of the difference of the circle area πR^2 and the area of the large triangle sqrt(3)b^2. That is-

$$A_{segment} = R^2\left[\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right]$$

There are numerous variations on this problem which can make the solution much more difficult. For instance, one could inscribe an odd sided regular polygon with five or more sides into a circle of radius R and then ask about the length segments of a chord drawn parallel to a line passing through the polygon center. This is not an easy problem to solve Even sided polygons on the other hand are much easier to treat because of symmetry. Here is a schematic for a square in a circle-



I leave it for the reader to show that this time the a to b ratio along the drawn chord is-

$$\frac{a}{b}=\frac{1}{2}\left\{\sqrt{1+\frac{4\Delta}{b}}-1\right\}$$

For the special case where Δ=b we recover the result b=aφ.

August 14, 2016