USE OF COMPLEX VARIABLES TO FIND ZEROS OF FUNCTIONS

It is well known that one can take any function of a complex variable \( F(z) \) and write it as a real part \( U(x,y) \) and an imaginary part \( V(x,y) \). From this result one can define a two variable continuous function-

\[
G(x, y) = \text{Abs}[F(z)] = \sqrt{U(x, y)^2 + V(x, y)^2}
\]

It represents the absolute value of \( F(z) \), so that whenever it vanishes we have a zero for \( G(\ x, y) \) in the \( x\)-\( y \) plane. This means one can always construct a function \( G(x, y) \) which will have zeros at specified points. It is our purpose here to generate several of such functions and to tie the results to other known unique curves when making contour plots.

Let us begin with a simple function \( G(x,y) \) which has zeros at \([-1,0]\) and \([1,0]\). Such a function follows from the complex variable function product

\[
F(z) = (z + 1)(z - 1) = z^2 - 1 = (x^2 - 1 - y^2) + 2xy
\]

It produces the function-

\[
G(x, y) = \sqrt{(x^2 + y^2)^2 - 2(x^2 - y^2) + 1}
\]

Using our MAPLE computer program we can conveniently plot this function in form of a contourmap using the two line program-

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with(plots): contourplot(G, x=-3..3,y=-2..2, contours=[1/4,1/2,1,2,4], grid=[100,100], color=blue, scaling=constrained);
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The resultant plot follows-
Note the symmetry about both the x and the y axis. This is expected since these two variables x and y enter the G(x,y) equation only as squares. The two zeros located at [±1,0] are confirmed. Also one has a saddle point at [0,0]. The infinity shaped figure corresponding to contour G(x,y)=1 is the famous lemniscate of Bernoulli. In Cartesian form it reads-

\[(x^2 + y^2)^2 = 2(x^2 - y^2)\]

and in polar form equals

\[r^2 = 2\cos(2\theta)\]

We can generalize the above example to a 2D function G(x,y) having N simple zeros. One looks at the corresponding complex variable function-

\[F(z) = \prod_{n=1}^{N} (z - z_n) = U(x,y) + iV(x,y)\]

The function G(x,y) now becomes-

\[G(x,y) = \sqrt{U(x,y)^2 + V(x,y)^2}\]

If the integer power of (z-z_n) is taken as greater than one one has a higher order zero.
Consider next the complex variable function-

\[ F(z) = (z^2 + 1)(z^2 - 1) = z^4 - 1 \]

It has four simple zeros at \([-1,0],[1,0],[0,1]\) and \([0,-1]\) The corresponding function \(G(x,y)\) is a 4th power polynomial in both \(x\) and \(y\). A contour plot of \(G(x,y)\) looks like this-

The contour \(G(x,y)=1\) forms a separatrix.

The function \(F(z)\) can take on a rather complicated forms yet one can still locate where the zeros of \(F(z)\) and \(\text{Abs}(F(z)) = G(x,y)\) are found. A beautiful example of this occurs in connection with the Riemann Zeta Function. Although it has not yet been proven to be correct for all points in the positive half of the \(z\) plane, Riemann postulated some 150 years ago that all zeros of the Zeta Function lie along a strip \(z=(1/2)+ib\) in the complex plane. The Riemann Zeta Function is defined as-

\[ \zeta(x + iy) = \sum_{n=1}^{\infty} \frac{1}{n^{x+iy}} \]

This function occurs often enough in the mathematical literature to be incorporated into most mathematics programs such as MAPLE and MATHEMATICA. We have-
U(x,y)=\text{Re}[\zeta(x+iy)] \quad \text{and} \quad V(x,y)=\text{Im}[\zeta(x+iy)]

and the result-

\[ G(x,y) = \text{Abs}[\zeta(x+iy)] \]

A contour plot of \( G(x,y) \) in the strip \(-1<x<3, \, 10<y<28\) produces the interesting pattern-

The results show that the first three zeros of the absolute value of the Zeta function lie along the line \( x=1/2 \) at \( y=14.13, \, 21.02, \) and \( 25.01 \) just as Riemann postulated. A good indication of what value of \( b \) corresponds to a zero is where the \( C=2 \) contour locally reaches its largest the negative \( x \) value.

Another complex function is-

\[ F(z) = \sin z = z \prod_{n=1}^{\infty} \left[ 1 - \left( \frac{z}{n \pi} \right)^2 \right] \]

It has zeros at \( z=0 \) and at \( z=\pm n \pi \). Thus we can expect the function-
\[ G(x,y) = \text{Abs}[\sin(z)] = \sqrt{\sin(x)^2 \cosh(y)^2 + \cos(x)^2 \sinh(y)^2} \]

to also have an infinite number of zeros lying along the x axis at integer values. A contourplot for \(G(x,y)\) follows-

![Contour plot of \(\text{Abs}[\sin(z)]\) showing zeros along the x-axis](image)

This cats eye pattern is reminiscent of that found for the phase plane plot for a simple undamped pendulum. The zeros are located at \(x=0, \pm 1, \pm 2,\) etc with the separatrix found at \(G(x,y)=1.\) Saddle points are found at \(\pi n/2\) for \(n=\pm 1, \pm 2,\) etc.

Another interesting contour map can be created from the identity-

\[ G(x, y) = \text{Abs}\{(z - \sqrt{3}/2 + i/2)(z - i)(z - \sqrt{3}/2 - i/2)\} \]

An expansion yields-

\[ G(x, y) = \sqrt{(x^6 + 3x^4y^2 + 3x^2y^4) + (1 - 2y^3 + 6x^2y + y^6)} \]

A contourplot yields the following pattern-
The function has three zeros located at the vertexes of an equilateral triangle centered at the origin and having side length $\sqrt{3}$. The contour $G(x,y)=1$ produces a curve reminiscent of the classical three-petal rose given in polar coordinates by
\[
r = a \cos[3(\theta + \pi/6)].
\]

In the area of fluid mechanics one encounters a complex variable representation which can represent both the velocity potential $\phi$ and streamfunction $\psi$ for an inviscid and incompressible 2D flow. The relation reads-
\[
F(z) = \phi + i\psi
\]
That is, some $F(z)$s can yield a streamline contour pattern –
\[
\psi(x,y) = \text{Im}[F(z)]
\]
\and the corresponding velocity potential pattern-
\[
\phi(x,y) = \text{Re}[F(z)]
\]
with the contours \( \psi = \text{const.} \) and \( \phi = \text{const.} \) forming an orthogonal set of curves. The orthogonality follows from the fact that \( F(z) \) satisfies the Cauchy-Riemann conditions.

Consider one of the best known of these complex velocity potentials-

\[
F(z) = z + \frac{1}{z} = (r + \frac{1}{r}) \cos(\theta) + i(r - \frac{1}{r}) \sin(\theta) = U(r, \theta) + iV(r, \theta)
\]

Having expressed this result in polar coordinates, where one sets \( z = re^{i\theta} \), yields a much simpler form than what would be achieved by using the Cartesian representation based upon \( z = x + iy \).

For such a fluid flow, consisting of a rectilinear flow and a doublet, it does not pay to look directly at-

\[
G(r, \theta) = \sqrt{(r^2 + \frac{1}{r^2}) + 2(2 \cos^2 \theta - 1)}
\]

Rather one should generate contour plots for \( U(r, \theta) \) and \( V(r, \theta) \). We have done this yielding the following flow pattern-

We have given the flow pattern only over the first quadrant of the x-y plane. The values in the remaining three quadrants can be gotten from this result by simple symmetry considerations about the x and y axis. In fluid mechanics this picture is referred to as...
invicid flow about a unit radius circle. A quarter of this circle is shown in black in the graph. The flow pattern inside the circle represents a doublet. As already mentioned there is no real physical meaning associated with $G(x,y)$ for this cylinder flow. Nevertheless let us briefly see what the pattern for

$$G(x,y) = \text{Abs}(z+1/z)$$

looks like. We find it has the following contourplot-

The pattern is quite interesting. For finite $[x,y]$, we have two simple zeros at $[0,\pm 1]$ and one infinity at $[0,0]$. If we treat the contour constant $C$ as a third variable one finds the 3D graph-
In all the above examples we have always gone from a complex function \( F(z) \) to the 2D function \( G(x, y) \). What about the reverse? In many instances this can be done although requiring more labour. Take for example the Lyapunov function-

\[
G(x, y) = x^2 + y^2
\]

The contour map for this function has contours corresponding to concentric circles centered on the origin. The contour \( C=0 \) is a point at the origin and indicates the location of this function’s zero. Writing things out we have-

\[
\left[ G(x, y) \right]^2 = (x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2 = U^2 + V^2
\]

Thus the corresponding complex function becomes –

\[
F(z) = (x^2 - y^2) + i(2xy) = z^2
\]

This also has a zero at \( z=x+iy=0 \).

Look next at the function-

\[
G(x, y) = \sqrt{(x^2 - y^2 + 1)^2 + (2xy)^2}
\]

there we have-

\[
F(z) = (x^2 - y^2 + 1) + i(2xy) = z^2 + 1
\]
with zeros at $z=+i$ and $z=-i$. The generation of $F(z)$ from a $G(x,y)$ will generally be more difficult than the reverse process where we are given a complex function $F(z)$ and then calculate its absolute value $G(x,y)$.

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