FINDING THE ZEROS OF THE ZETA FUNCTION USING CONTOUR PLOTS

It is well known that the Zeta Function is defined as the infinite sum -

\[ \zeta(x + iy) = \sum_{n=1}^{\infty} \frac{1}{n^{x+iy}} = \sum_{n=1}^{\infty} \frac{1}{n^x} \{ \cos[y \ln(x)] - i \sin[y \ln(x)] \} \]

From this definition we can define two new functions-

\[ U(x, y) = \text{Re}[\zeta(x + iy)] \quad \text{and} \quad V(x, y) = \text{Im}[\zeta(x + iy)] \]

With them one finds the absolute value of Zeta to be-

\[ F(x, y) = \sqrt{U(x, y)^2 + V(x, y)^2} \]

Here \( F(x, y) \) is always real and vanishes only at zeros of the Zeta Function.
From the above definition of \( \zeta(x + iy) \) we note that it is anti-symmetric about the x axis relative to y. That is, \( \zeta(x + iy) = a + ib \) while \( \zeta(x - iy) = a - ib \).
It is not symmetric about the y axis. We also find-

\[ \zeta(-1+2i) = 0.1689156698 - i0.07051598891 \]

and-

\[ F(-1, 2) = \text{Abs}[\zeta(-1+2i)] = 0.1830437330 \]

At first glance, the last two results seem to not be correct considering that the \( 1/n^x \) term in the Zeta Function definition implies an infinity whenever x becomes negative. That this is not so must be due to the remaining sine and cosine terms in the sum.

We next come to one of the most important properties of the Zeta Function, namely, the location of its zeros. A new way to establish their location is via contour-maps. To create such contour-maps is a relatively easy task and involves the simple one line computer command-

```
contourplot(F, x=1..3, y=10..34, thickness=1, grid=[50,150], contours=[0.5,1,2], color=blue, scaling=constrained);
```

This produces the following result-
What is most interesting about this result it that it indicates that the only zeros possible lie within the indicated circles. That is the zeros of the Zeta Function, when they exist, must lie within the semi-infinite strip-

$$0.1 < x < 0.9 \quad \text{and only for certain values of } y$$

Any point outside this strip has contour values greater than $C=0.5$ meaning no possibility of zeros. It was the German mathematician Bernhard Riemann who in the late 1860s first proposed his famous hypothesis which states that-

**All non-trivial zeros of the Zeta Function have a real part of one-half**

With the aide of electronic computer, the first several thousand zeros of the Zeta Function have now been found and they indeed all fall along the line $x=1/2$. However, a general proof of the hypothesis still awaits. The above contour map clearly shows that the Zeta zeros fall in the centers of the circles shown and that these centers correspond to $x=1/2$.

To get a better feel for the $x$ location of the Zeta zeroes we next magnify our contour-plot about the first zero near $x=1/2$ and $y=14.1$ using the modified computer command-

```
contourplot(F,x=0.4..0.8,y=14.00..14.24,contours=[0.02,0.04,0.08],color=black);
```

Its application produces the following contour-plot-
We see that the contours are essentially concentric circles with the zero lying within the inner circle. This means, also when looking at subsequent zeros, that all zeros of the Zeta Function lie in the even narrower semi-infinite strip:

$$0.48 < x < 0.52 \text{ with } y > 0$$

Continuing on to ever higher magnifications we have that all zeros indeed lie along the line $$x=1/2$$ as the Riemann Hypothesis suggests.

The final step of determining the first five zeros of the Zeta Function simply involves looking at the real part of $$\zeta(1/2+iy)$$ near the suggested y values from the first of the above contour plots. Near the first zero this produces:

$$\zeta(1/2+i14.1) = 0.004698400183 - i0.02705828237$$

and:

$$\zeta(1/2+i14.2i) = -0.006816218159 + i0.05159699098$$

The sign change for the real part of the Zeta function in these values tells us that the value of y for the first zero lies between 14.1 and 14.2. So we next look at:

$$\text{Re}[\zeta(1/2+i14.13)] = 0.0005960776782 \quad \text{and Re}[\zeta(1/2+i14.14)] = -0.0006492183866$$

This indicates the zero lies between $$y=14.13$$ and $$y=14.14$$. Continuing this procedure on one eventually reaches the result that the first zero of the Zeta function lies at:

$$[x,y]=[1/2, 14.134725\ldots]$$

Working out the location of the zeros within the remaining four circles of the first contour-plot above using the same approach of sign change, we get:
2\textsuperscript{nd} zero \hspace{1em} [x,y]=[1/2, 21.022040] \\
3\textsuperscript{rd} zero \hspace{1em} [x,y]=[1/2, 25.010856] \\
4\textsuperscript{th} zero \hspace{1em} [x,y]=[1/2, 30.428762] \\
5\textsuperscript{th} zero \hspace{1em} [x,y]=[1/2, 33.623793] \\

Note that there is no obvious regularity in this sequence of zeros.

The above discussion has shown, via contour plotting, that all zeros of the Zeta Function $\zeta(x,y)$ indeed lie along the $x=1/2$ line in the complex $x$-$y$ plane and that the precise zero location for the $y$ values follow by a procedure which notices the sign change of the real part of $\zeta(1/2+iy)$ at the correct values of $y$.

U.H.Kurzweg  
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