N! AND THE GAMMA FUNCTION

Consider the product of the first n positive integers-

$$1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot \;\; \cdot(n-1)\cdot n=n!$$

One calls this product the n factorial and has that product of the first five integers equals 5!=120. Directly related to the discrete n! function one has the continuous gamma function Γ(n). Both are defined by the same integral –

$$\Gamma(n+1)=n!=\int_{t=0}^{\infty} t^n \exp(-t)dt$$

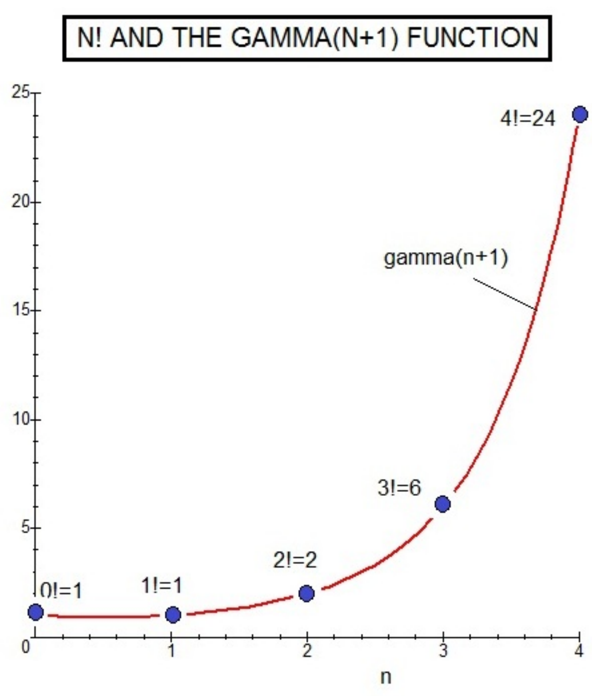
One integration by parts yields-

$$\int_{t=0}^{\infty} t^n \exp(-t)dt = n\int_{t=0}^{\infty} t^{n-1} \exp(-t)dt$$

from which follow the identities-

$$(n+1)!=n!(n+1) \qquad \text{and} \qquad \Gamma(n+1)=n\Gamma(n)$$

We also have that 0!=1!=1 and Γ(0)=∞, Γ(1/2)=sqrt(π) and Γ(1)=1. A plot of n! and Γ(n+1) follow-



The blue dots are n! values while the red curve represents the continuous Γ(n+1) function. The curve reaches a minimum value of Γ(n)= 0.88560 at n=1.462. One typically finds the values for Γ(n) are tabulated only in the range $1<n<2$, since the rest can be quickly generated via the above recurrence formula.
Although there is no true factorial for negative integers, one can extend the Γ(n) function to negative n and obtain non infinite values when n is a negative non-integer.

We can also express the derivative of the gamma function as the integral-

$$\frac{d\Gamma(n)}{dn} = \int_{t=0}^{\infty} \frac{\exp(-t)}{t}\left(\frac{dt^n}{dn}\right)dt = \int_{t=0}^{\infty} t^{n-1}\ln(t)\exp(-t)dt$$

This derivative goes toward -∞ as n->0, has zero value near n=1.462 and takes on progressively larger positive values as n heads toward plus infinity. Notice that this last integral is just the Laplace transform of $t^{(n-1)}\ln(t)$ after s is set equal to unity. A function related to Γ(x) is the digamma function-

$$\psi(x) = \frac{d\Gamma(x)/dx}{\Gamma(x)} = \frac{d[\ln\Gamma(x)]}{dx}$$

It has the value ψ(1)=-γ where γ=0.5772156649.. is the Euler constant.

One can also sum the reciprocals of various combinations of n!. We have, among many other examples, that-

$$\exp(1) = \sum_{n=0}^{\infty}\frac{1}{n!} = 2.718281828459045..$$

$$I_0(2) = \sum_{n=0}^{\infty}\frac{1}{(n!)^2} = 2.27958530233607..$$

$$\cosh(1) = \sum_{n=0}^{\infty}\frac{1}{(2n)!} = 1.54308063481524..$$

$$\sinh(1) = \sum_{n=0}^{\infty}\frac{1}{(2n+1)} = 1.17520119364380..$$

Notice also that-

$$(2n)!=[1\cdot3\cdot5\cdot\ (2n-1)][2\cdot4\cdot6\cdot\ 2n]= [1\cdot3\cdot5\cdot\ (2n-1)]2^n n!$$

Thus we have that-

$$1\cdot3\cdot5\cdot\ \cdot(2n-1)=(2n)!/(2^n n!)$$

From this it follows that the product of the first five odd numbers equals 10!/(32*5!)=945. This last form also allows one to write certain infinite series in compact form. For example, we have that-

$$\frac{1}{\sqrt{1-x^2}}=1+\frac{1}{2\cdot 1!}x^2+\frac{(1\cdot 3)}{2^2\cdot 2!}x^4+\frac{(1\cdot 3\cdot 5)}{2^3\cdot 3!}x^6+...=\sum_{m=0}^{\infty}\frac{(2m)!}{2^{2m}(m!)^2}x^{2m}+$$

As we learned in our earlier discussions on Legendre polynomials $P_n(x)$, these can be generated by the generating function -

$$\frac{1}{\sqrt{1+t^2-2xt}}=\sum_{n=0}^{\infty}P_n(x)t^n$$

So on setting $\varepsilon=2xt-t^2$ we can write-

$$\sum_{n=0}^{\infty}P_n(x)t^n=\sum_{m=0}^{\infty}\frac{(2m)!}{2^{2m}(m!)^2}[t(2x-t)]^m=1+\frac{2!}{2^2(1!)^2}[t(2x-t)]+\frac{4!}{2^4(2!)^2}[t(2x-t)]^2+...$$
$$=1+xt+[(3x^2-1)/2]t^2+O(t^3)$$

which produces the Legendre polynomials.

One can also use the gamma function Γ(n) to evaluate 1·3·5· ·2n-1. We have Γ(1/2)=sqrt(π) so that Γ(3/2)=sqrt(π)/2 , Γ(5/2)=(1·3)sqrt(π)/2^2 and Γ(7/2)=(1·3·5)sqrt(π)/2^3. From this it follows that-

[1·3·5·   ·(2n-1)]= 2^n Γ(n+1/2)/sqrt(π)

Combining this result with the form of (2n)! given earlier, one obtains the Legendre Duplication Formula-

$$(2n)!=\frac{2^{2n}n!}{\sqrt{\pi}}\Gamma(n+\frac{1}{2})$$

Trying this out for n=5, we find-

10!=(2^{10}5!)Γ(11/2)/sqrt(π)=3628800

Another combination of factorials which often arises is the famous binomial coefficient-

C_{nm}=n!/[(m!(n-m)!]

It is produced by the following binomial expansion-

$$(a+b)^n = a^n + na^{n-1}b/1! + n(n-1)a^{n-2}b^2/2! + = \sum_{m=0}^{n} C_{nm} a^m b^{n-m}$$

Note that the C_{nm} for a fixed n just represents the numbers in the nth row of a Pascal triangle. Thus the 4th row has the coefficients C_{4m}=4!/[(m!(4-m)!] which are 1-4-6-4-1.

Another extension of the factorial is the product of squares which read-

$$F(n)= 1\cdot4\cdot9\cdot16\cdot \quad \cdot n^2$$

This is easy to evaluate by noting F(n) is just the product of n! with itself. That is-

$$F(n)=(1\cdot2\cdot3\cdot \quad \cdot n)(1\cdot2\cdot3\cdot \quad \cdot n)=(n!)^2$$

It also follows that the product of the first n pth powers of the integers equals $(n!)^p$. Thus –

$$1\cdot8\cdot27\cdot64\cdot125=(5!)^3=1728000$$

Next we examine the value of Γ(n+1/2). Using the Legendre Duplication Formula and the form for (2n)! given earlier, we have-

$$\Gamma(n+\frac{1}{2}) = \frac{\sqrt{\pi}(2n)!}{2^{2n} n!}$$

This allows one to find the half-integer gamma function. It says –

$$\Gamma(31/2)=30!\mathrm{sqrt}(\pi)/(2^{30}15!)= (6190283353629375/32768)\mathrm{sqrt}(\pi)$$

As expected this value lies between 14! And 15!.

It is also possible to develop gamma function identities not found in existing mathematical handbooks. One of these is-

$$G(n)=\Gamma(n+1/2)\cdot\Gamma(n-1/2)$$

We develop the general value for G(n) by starting with n=1 where G(1)=π/2. Next at n=2 we have G(2)=+3π/8 and at n=3 we find G(3)=45π/32. This suggests that –

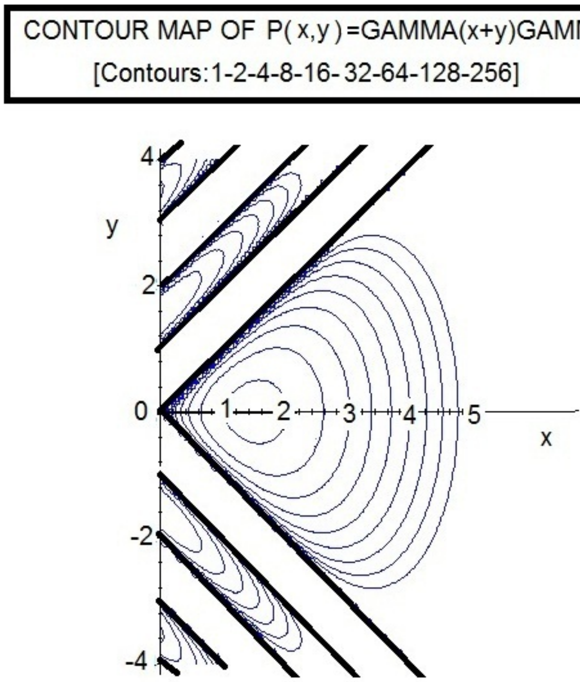
$$G(n) = \frac{(2n-1)\pi}{2^{2n-1}}\{1^2\cdot3^2\cdot5^2\cdot..\cdot(2n-3)^2\} = \frac{(2n-1)}{2}[\Gamma(n-\frac{1}{2})]^2$$

This identity checks for all values of n tried for n of one or greater.

Another variation is the gamma product function-

$$P(x,y)=\Gamma(x+y)\cdot\Gamma(x-y)$$

which reduces to $(x+y-1)!\cdot(x-y-1)!$ when x and y are integers. A contour plot of this function for $x>0$ and $-4<y<4$ looks like this-



The contours form closed curves and P(x,y) goes to infinity when $y=\pm(x+n)$ since gamma for any negative integer in unbounded. Alternate strip regions between the unit slope curves also show finite valued contours. The minimum contour value occurs near x=1.46 and y=0 and has the value P=0.7844.

Finally we look at the gamma function $\Gamma(z)$ when z=x+iy is a complex number. Here the best approach is to use the integral definition-

$$\Gamma(x+iy)=\int_{t=0}^{\infty} t^{x+iy-1}\exp(-t)dt=\int_{t=0}^{\infty} t^{x-1}\exp(-t)\left[\cos(y\ln t)+i\sin(y\ln t)\right]dt$$

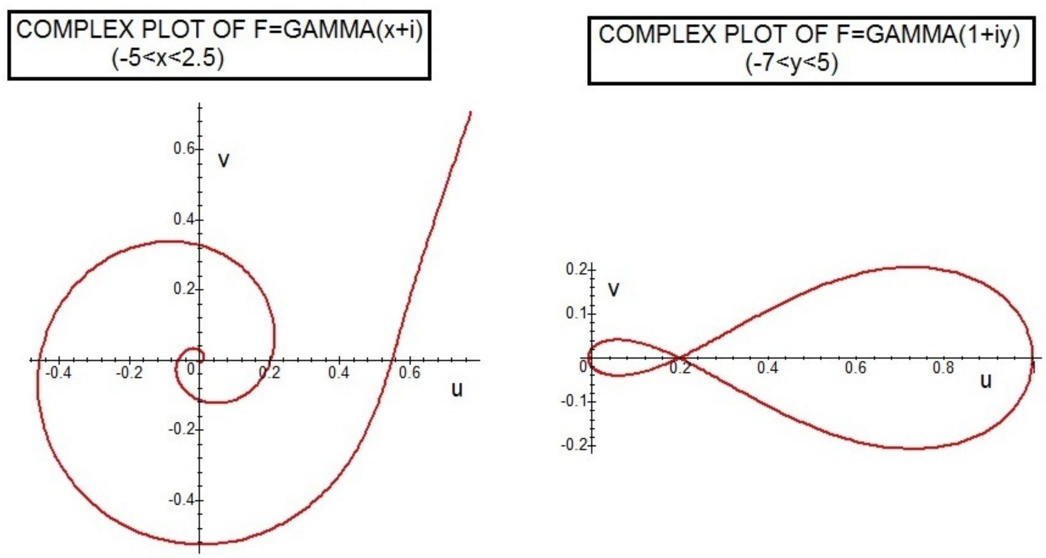
From the integral we see that $\Gamma(x+iy)$ has a real and imaginary part represented by two different integrals. We find-

$$\Gamma(1+i)= 0.4980156681-i0.1549498283$$

so that $\Gamma(1+i)\cdot\Gamma(1-i)=|\Gamma(1+i)|^2=0.2720290550..$. Also we have that-

$$\int_{t=0}^{\infty} \cos[\ln(t)]\exp(-t))dt = 0.4980156681..$$

We can also plot Γ(x+iy)=u+iv in the u-v plane. This can produce some interesting figures such as the following-



u=Real(F) v=Imag(F)

In the first we plot Γ(z) for z=x+i to get a run-away spiral pattern. For the second figure we have set z=1+iy. It produces a closed double loop. Many other plots are possible by just setting x or y to different constant values.

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