Sample Problems for Exam II

1. The shaft below has length L, Torsional stiffness GJ and torque T is applied at point C, which is at a distance of 0.6L from the left (point A). Use Castigliano theorem to Calculate the maximum twist angle.

   ![Shaft Diagram]

   **Solution:** Denote the reaction torque applied at point B as TB. Then the torque in the segment BC is equal to T_B, and the torque in segment AC is equal to T+T_B. The Strain energy is 
   \[ U = \frac{(T+T_B)^2(0.6L)}{2GJ} + \frac{T_B^2(0.4L)}{2GJ} \]

   We find T_B by differentiating U with respect to T_B and setting the derivative to zero (because the twist angle at point B is zero)
   \[ \frac{\partial U}{\partial T_B} = \frac{(T+T_B)(0.6L) + T_B(0.4L)}{GJ} = 0 \Rightarrow T_B = -0.6T \]

   We now substitute T_B into the energy expression and differentiate with respect to T to get the twist angle at C, which is the maximum twist angle.
   \[ U = \frac{(0.4T)^2(0.6L)}{2GJ} + \frac{(-0.6T)^2(0.4L)}{2GJ} = \frac{0.12T^2L}{GJ} \]
   \[ \phi_C = \frac{\partial U}{\partial T} = \frac{0.24TL}{GJ} \]

2. A thin aluminum sheet is to be used to form a closed thin-walled section. If the total length of the wall contour is 100 cm, what is the shape that would achieve the highest torsional rigidity? Consider elliptical (including circular), rectangular, and equilateral triangular shapes.

   **Solution:**
   Torsional rigidity is
   \[ J = \frac{4A^2}{\oint ds/t} \]  
   (1)

   In this problem t is constant and same for all shapes. Also \( \oint ds = 100cm \) fixed for all the shapes. Therefore shape of largest A would give the highest rigidity.

   *Rectangular of 2(a + b) = 100*
\[
\overline{A} = ab
\]

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>\overline{A}</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>40</td>
<td>400</td>
</tr>
<tr>
<td>15</td>
<td>35</td>
<td>525</td>
</tr>
<tr>
<td>25</td>
<td>25</td>
<td>625</td>
</tr>
</tbody>
</table>

Equilateral triangular of 3a=100

\[
\overline{A} = \frac{a^2}{2} \sin 60 = 481.125
\]

Elliptical section

\[
\text{perimeter} = 100 \approx \pi \sqrt{2(a^2 + b^2) - (a - b)^2 / 2}
\]

\[
\overline{A} = \pi ab
\]
<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>$\bar{A}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>20.88</td>
<td>656</td>
</tr>
<tr>
<td>15</td>
<td>16.8</td>
<td>792</td>
</tr>
<tr>
<td>15.92</td>
<td>15.92</td>
<td>796</td>
</tr>
</tbody>
</table>

$a=b=15.92$ is a circular section and offers the highest rigidity.

3. This is a part of Problem 6.65. A solid circular shaft has a diameter of 200mm. A second shaft has the same elastic shear strength but is hollow with an inside diameter of 150mm. Both shafts are torqued to their shear strength by torques of equal magnitude. This yields an outer diameter of the hollow shaft of 217.75mm. Find the ratio of the fully plastic torques of the two shafts for elastic-perfectly plastic material.

**Solution:** For the solid shaft $T_{fp} = \frac{\pi}{12} \tau_y D^3$ while for the hollow shaft

$$T_{fp} = \frac{\pi}{12} \tau_y \left( D_o^3 - D_i^3 \right)^{\frac{3}{2}}$$

so the ratio is $r = \frac{D^3}{D_o^3 - D_i^3} = \frac{200^3}{217.75^3 - 150^3} = 1.15$

4. The stringer-web sections shown in figures are subject to shear $V_z \neq 0$, while $V_y = 0$. Find the bending stress in the stringers for the same bending moment $M_y$. Which section is most effective in bending?

**Solution:** (solution is in a coordinate system with $x$ being the beam axis, and $y$-$z$ being the coordinates in the plane of the cross section.)

(a)

Given, $V_z \neq 0 \rightarrow M_y \neq 0$

$I_y \neq 0$,

$I_{yz} = 0$ (symmetry)

Axial stress due to Moment $M_y$

$$\sigma_{xx} = \frac{M_y h}{I_y}$$

For the section (a),

$I_y = 2Ah^2 + 2Ah^2 = 4Ah^2$. Therefore
\[ \sigma_{xx} = \frac{M_y}{4Ah} \] (1)

(b)

\[ I_y \neq 0 \]
\[ I_{yz} = 0 \text{ (Section is symmetric with respect to y)} \]
\[ I_y = Ah^2 + Ah^2 + Ah^2 + Ah^2 = 4Ah^2 \]. Then,
\[ \sigma_{xx} = \frac{M_yh}{I_h} = \frac{M_y}{4Ah} \] (2)

(c)

\[ I_{yz} = \sum A_i y_i z_i \]
\[ = A(-h)(h) + A(0)(h) + A(0)(-h) + A(h)(-h) = -2Ah^2 \]
\[ I_y = \sum A_i z_i^2 = 4Ah^2 \]
\[ I_z = \sum A_i y_i^2 = 2Ah^2 \]

Let's find the centroid of the section (origin of the yz system),
\[ \bar{y}_c = \frac{\sum y_i A_i}{\sum A_i} = \frac{-hA + 0 + 0 + hA}{4A} = 0 \]
\[ \bar{z}_c = 0 \]
So, \( \bar{y} \) & \( \bar{z} \) is at centroid and \( \bar{y} \bar{z} \equiv yz \).
From equation 4.29 (with $M_z = 0$)

\[
\sigma_{xx} = -\frac{I_{yz}M_Y}{I_yI_z - I_{yz}^2} y + \frac{I_zM_Y}{I_yI_z - I_{yz}^2} z
\]

\[
\sigma_{xx} = M_Y \left[ \frac{2Ah^2}{8A^2h^4 - 4A^2h^4} y + \frac{2Ah^2}{8A^2h^4 - 4A^2h^4} z \right]
\]

\[
= \left( \frac{M_y}{Ah^2} \right) \left( \frac{y}{2} + \frac{z}{2} \right)
\]  

(3)

Neutral Axis for the section (c) and moment $M_y$ can be found as

\[
\tan \alpha = \frac{-I_{yz}M_Y}{I_zM_Y} = 1 \rightarrow \alpha = 45^\circ.
\]

If we plot this on the section,

Magnitude of the stresses (absolute value of the stress) will be identical at points 2 and 3. Stresses at 1 and 4 are zero since the neutral axis passes through them. Therefore it is sufficient to check points 2 and 3 for the maximum magnitude of the stresses.

Axial stress at points (1) & (4) \( \rightarrow \sigma_{xx} = 0 \)

Axial stress at points (2) & (3) \( \rightarrow |\sigma_{xx}| = \frac{1}{2} \frac{M_y}{Ah} \)  

(4)

Comparing (1), (2) and (4) we can conclude that:

- Section (b) and (a) has same effectiveness for bending.
- Section (c) is the least effective as the $\sigma_{xx}$ is the largest although same amount of material is used in all (a), (b) and (c).

5. What are the equations for the in-plane displacements in torsion of a circular cross section? How do we see that they correspond to rigid body rotation of the section? What is different about the displacements in a non-circular cross-section?
The displacements are: \[u = -y \beta \theta z, \quad v = x \beta \theta z\]. They correspond to zero strains in the plane of the cross-section: \[\varepsilon_{xx} = \frac{\partial u}{\partial x} = 0, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y} = 0, \quad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0\]. For a non-circular cross section, \(u\) and \(v\) are still the same, but \(w\) is not zero (warping).

6. Consider an equilateral triangle and a bending moment that in the elastic ranges produces maximum tensile stresses at a vertex and maximum compressive stresses on the side opposite this vertex. If the material is elastic-perfectly plastic, where is the neutral axis for the fully plastic moment?

**Answer:** It is parallel to the side which had the maximum compressive stresses, and it divides the triangle into two equal areas. Since the area of the triangular side is proportional to the square of the length of the sides, these have to be reduced by square root of 2.

7. When are the Euler Bernoulli assumptions invalid for slender beams?

**Answer:** When the Shear modulus is much smaller than Young’s modulus.