Energy and complementary energy

- In linear range $U = C$
Virtual displacements and generalized coordinates

- System with \( n \) degrees of freedom under virtual displacement
  \[
  (x_1, x_2, \ldots, x_n) \Rightarrow (x_1 + \delta x_1, x_2 + \delta x_2, \ldots, x_n + \delta x_n)
  \]

- Generalized coordinates will typically be distances or angles

- Generalized forces are defined as the complementary quantity that when multiplied by generalized displacement gives the resulting work
  \[
  \delta W = Q_1 \delta x_1 + Q_2 \delta x_2 + \cdots + Q_n \delta x_n
  \]
Conservative system

• Internal and external virtual work

\[ \delta W = \delta W_e + \delta W_i \]

\[ \delta W_i = -\delta U = \frac{\partial U}{\partial x_1} \delta x_1 + \frac{\partial U}{\partial x_2} \delta x_2 + \cdots + \frac{\partial U}{\partial x_n} \delta x_n \]

\[ \delta W_e = P_1 \delta x_1 + P_2 \delta x_2 + \cdots + P_n \delta x_n \]

• Altogether

\[ Q_1 \delta x_1 + Q_2 \delta x_2 + \cdots + Q_n \delta x_n = P_1 \delta x_1 + P_2 \delta x_2 + \cdots + P_n \delta x_n \]

\[ -\frac{\partial U}{\partial x_1} \delta x_1 - \frac{\partial U}{\partial x_2} \delta x_2 + \cdots - \frac{\partial U}{\partial x_n} \delta x_n \]

\[ Q_i = P_i - \frac{\partial U}{\partial x_i} \]
Principle of stationary potential energy

- In equilibrium, $\delta W = 0$ known as principle of stationary potential energy.
- Then $P_i = \frac{\partial U}{\partial x_i}$
- Known as Castigliano’s first theorem (Carlo Alberto Castiglino, 1847-1884)
In an element with a nonuniform stress distribution

\[ u = \lim_{\Delta V \to 0} \frac{\Delta U}{\Delta V} = \frac{dU}{dV} \quad U = \int u \, dV = \text{total strain energy} \]

\[ U = \int \frac{\sigma_x^2}{2E} \, dV = \int \frac{P^2}{2AE} \, dx \quad \text{since} \quad dV = A \, dx \]

For a rod of uniform cross-section, under axial loading,

\[ \sigma_x = \frac{P}{A}, \quad dV = A \, dx \quad \rightarrow \quad U = \frac{P^2L}{2AE} \]
Three-bar truss example

• Strains and stresses
  \[ \varepsilon_B = \frac{v}{L} \quad \varepsilon_A = \frac{(v + \sqrt{3}u)}{4L} \quad \varepsilon_C = \frac{(v - \sqrt{3}u)}{4L} \]

• Strain energy
  \[ n_B = \frac{EA}{L}v \quad n_A = \frac{E(v + \sqrt{3}u)}{4L} \quad n_C = \frac{E(v - \sqrt{3}u)}{4L} \]

\[ U = \left( \frac{EA}{L}v \right)^2 \frac{L}{2AE} + \left( \frac{E(v + \sqrt{3}u)}{4L} \right)^2 \frac{(2L)}{2AE} + \left( \frac{E(v - \sqrt{3}u)}{4L} \right)^2 \frac{(2L)}{2AE} = \frac{EA(1.25v^2 + 0.75u^2)}{2L} \]

• Equilibrium
  \[ \frac{\partial U}{\partial u} = 0.75 \frac{EA}{L} u = p \quad \frac{\partial U}{\partial v} = 1.25 \frac{EA}{L} v = p \]
Castigliano’s (second theorem)

• If an elastic system is supported so that rigid-body displacements of the system are prevented, and if certain concentrated forces of magnitude $F_1$, $F_2$, ..., $F_p$ act on the system, in addition to distributed loads and thermal strains, the displacement component $q_i$ of the point of application of the force $F_i$ is determined by the equation

$$q_i = \frac{\partial C}{\partial F_i}$$

• How do we calculate $C$?
Beam in bending

\[ \sigma_x = \frac{My}{I} \rightarrow U = \int \frac{\sigma_x^2}{2E} \, dV = \int \frac{M^2 y^2}{2EI^2} \, dV \quad dV = A \, dx \]

\[ U = \int_0^L \int_A \frac{M^2 y^2}{2EI^2} \, dA \, dx = \int_0^L \frac{M^2}{2EI} \left( \int_A y^2 \, dA \right) \, dx = \int_0^L \frac{M^2}{2EI} \, dx \]
Simplest example

- What is the rotation of the end of a cantilever beam under an end moment $M$?
- Strain energy equal to complementary strain energy

$$C = U = \frac{M^2 L}{2EI}$$

- Rotation at tip

$$\theta = \frac{\partial C}{\partial M} = \frac{ML}{EI}$$

- What do we do if we want displacement at tip?
Shear strain energy

\[ u = \int_0^{\gamma_{xy}} \tau_{xy} \, d\gamma_{xy} \quad \text{shear in plane stress} \]

Within the proportional limit:

\[ u = \frac{1}{2} G \gamma_{xy}^2 = \frac{1}{2} \tau_{xy} \gamma_{xy} = \frac{\tau_{xy}^2}{2G} \]

The total strain energy: \( U = \int u \, dV = \int \frac{\tau_{xy}^2}{2G} \, dV \)

For a shaft subjected to a torsional load,

\[ U = \int \frac{\tau_{xy}^2}{2G} \, dV \]

Since \( \tau_{xy} = \frac{T \rho}{J} \rightarrow U = \int \frac{T^2 \rho^2}{2GJ^2} \, dV \quad dV = dA \, dx \),

\[ U = \int \int \frac{T^2 \rho^2}{2GJ^2} \, dA \, dx = \int \frac{T^2}{2GJ^2} \left( \int A \rho^2 \, dA \right) dx = \int_0^L \frac{T^2}{2GJ} dx \]

Uniform shaft, \( U = \frac{T^2 L}{2GJ} \)
Example

- The polar moment of inertia of a shaft is reduced by a factor of two from root to tip as \( J = J_0 / (1 + x/L) \). Calculate the percent reduction in weight and percent increase in twist angle at the tip. Compare to uniform shaft of the same volume.
- **Area**: The area is proportional to the square of the radius while the moment of inertia is proportional to the 4\(^{th}\) power. So the area at the tip is 70.7% of the area at the root. Integrating, we find that the volume is reduced by 17.2%.
- **Twist angle**: 1/J increases linearly to double at the tip, so the twist angle increases by 50%.
- For **uniform shaft**, a reduction in volume of 17.2% will reduce J to \((1-0.172)^2 = 0.686J_0\), leading to a twist angle increase by \(1/0.686 = 1.46\), only 46%.
Transverse shear

\[ U = \int \frac{\tau^2}{2G} dV = \int \frac{1}{2G} \left( \frac{VQ}{It} \right)^2 dAdx \]

\[ = \int_0^L \frac{V^2}{2GI^2} \left( \int_A \frac{Q^2}{t^2} dA \right) dx \]

\[ = \int_0^L \frac{V^2}{2GA} \left( \frac{A}{I^2} \int_A \frac{Q^2}{t^2} dA \right) dx \]

\[ = \int_0^L \frac{kV^2 dx}{2GA}, \quad \text{where } k = \text{correction factor for shear} \]

<table>
<thead>
<tr>
<th>Cross Section</th>
<th>k</th>
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<tr>
<td>Rectangle</td>
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Reading assignment

Section 5.3-4: Question: What is “dummy” about “dummy load method?”

Source: www.library.veryhelpful.co.uk/Page11.htm