5.3 Castigliano’s theorem on deflection for linear load-deflection relations

- For this case complementary strain energy is equal to strain energy and we get

\[ q_j = x_j = \frac{\partial U}{\partial P_j} \quad \text{and} \quad \theta_j = \frac{\partial U}{\partial M_j} \quad \phi_j = \frac{\partial U}{\partial T_j} \]

- For general case

\[
x_i = \frac{\partial U}{\partial F_i} = \int \frac{N}{EA} \frac{\partial N}{\partial F_i} \, dx + \int \frac{kV}{GA} \frac{\partial V}{\partial F_i} \, dx + \int \frac{M}{EI} \frac{\partial M}{\partial F_i} \, dx + \int \frac{T}{GJ} \frac{\partial T}{\partial F_i} \, dx
\]

\[
\theta_i = \frac{\partial U}{\partial M_i} = \int \frac{N}{EA} \frac{\partial N}{\partial M_i} \, dx + \int \frac{kV}{GA} \frac{\partial V}{\partial M_i} \, dx + \int \frac{M}{EI} \frac{\partial M}{\partial M_i} \, dx + \int \frac{T}{GJ} \frac{\partial T}{\partial M_i} \, dx
\]

\[
\phi_i = \frac{\partial U}{\partial T_i} = \int \frac{N}{EA} \frac{\partial N}{\partial T_i} \, dx + \int \frac{kV}{GA} \frac{\partial V}{\partial T_i} \, dx + \int \frac{M}{EI} \frac{\partial M}{\partial T_i} \, dx + \int \frac{T}{GJ} \frac{\partial T}{\partial T_i} \, dx
\]
Procedure

• Write an expression for each of the internal actions (axial force, bending moment, shear force, and torque) in each member of the structure in terms of external loads.

• Take derivatives of strain energy to get deflections and/or rotations
Example

The cantilever beam $AB$ supports a uniformly distributed load $w$ and a concentrated load $P$ as shown (Fig. 10.40). Knowing that $L = 2 \text{ m}$, $w = 4 \text{ kN/m}$, $P = 6 \text{ kN}$, and $EI = 5 \text{ MN} \cdot \text{m}^2$, determine the deflection at $A$.

\[
M = -(Px + 0.5wx^2) \quad \frac{\partial M}{\partial P} = -x
\]

\[
y_A = \int_0^L \frac{M}{EI} \frac{\partial M}{\partial P} dx = \frac{1}{EI} \int_0^L (Px^2 + 0.5wx^3) dx = \frac{1}{EI} \left( \frac{PL^3}{3} + \frac{wL^4}{8} \right)
\]

- What displacement will we get if we differentiate with respect to $w$?
Problem 3: The frame ABCD is in the horizontal plane. One end of the frame (A) is fixed to a rigid wall and the other end (D) is subjected to a vertical force $F$.

(a) Use Castigliano’s Theorem to determine the vertical deflection at D.
(b) What is the strain energy stored in the frame?

Consider only strain energies due to bending and torsion. Length $AB=BC=CD=L$. Flexural and torsional rigidities of all members can be taken as $EI$ and $GJ$, respectively.

Some useful formulas
Strain energy per unit length in flexure = $M^2/2EI$
Strain energy per unit length in torsion = $T^2/2GJ$

Caution: Note that ABCD is in the horizontal plane perpendicular to the direction of $F$!
Solution

- Member CD acts like cantilever beam with end load

\[ M = Fx \]

\[ U_{CD} = \int_0^L \frac{M^2}{2EI} \, dx = \frac{F^2}{6EI} L^3 \]

- Member BC has in addition torque FL

\[ U_{BC} = \frac{T^2 L}{2GJ} + \frac{F^2 L^3}{6EI} = \frac{F^2 L^3}{2GJ} + \frac{F^2 L^3}{6EI} \]

- Member AB has the end force plus a clockwise moment FL plus torque FL

\[ M = Fx - FL \]

\[ U_{AB} = \frac{F^2 L^3}{3EI} + \frac{F^2 L^3}{2GJ} \]

- Altogether

\[ U = \frac{2F^2 L^3}{3EI} + \frac{F^2 L^3}{GJ} \]

\[ y_D = \frac{\partial U}{\partial F} = \frac{4FL^3}{3EI} + \frac{2FL^3}{GJ} \]

How can we solve this problem as easily without castigliano?
Dummy load method

- When we want to calculate displacements or rotation at a point where no forces are applied use the following procedure
  1. Apply a “dummy” load at the point
  2. Calculate the energy in term of actual forces plus dummy load
  3. Take derivative with respect to magnitude of dummy load
  4. Set magnitude of dummy load to zero
Example

- What is the rotation at point D?
- Add clockwise moment $m$ at point D
- Examine contribution of member CD

$$M = Fx + m \quad U_{CD} = \int_0^L \frac{(Fx + m)^2}{2EI} \, dx \quad \frac{\partial U_{CD}}{\partial m} \bigg|_{m=0} = \int_0^L \frac{Fx}{EI} \, dx = \frac{FL^2}{2EI}$$

- Member BC will have the same contribution in bending and in addition a torque $m$

$$\left( U_{BC} \right)_{\text{torsion}} = \frac{(T + m)^2 L}{2GJ} \quad \frac{\partial \left( U_{BC} \right)_{\text{torsion}}}{\partial m} \bigg|_{m=0} = \frac{TL}{GJ} = \frac{FL^2}{GJ}$$
Example continued

- Member AB

\[ M_{AB} = Fx - (FL - m) \]

\[ (U_{AB})_{bending} = \int_{0}^{L} \frac{(Fx + m - FL)^2}{2EI} \, dx \]

\[ \frac{\partial (U_{AB})_{bending}}{\partial m} \bigg|_{m=0} = \int_{0}^{L} \frac{(Fx - FL)}{EI} \, dx = -\frac{FL^2}{2EI} \]

- Altogether

\[ \theta_D = \frac{FL^2}{GJ} \]

- Please check!
Maxwell Reciprocal Theorem
(James Clerk Maxwell 1831-1879)

- If we have two generalized loads $Q_1$ and $Q_2$, strain energy will be a quadratic form in both
  \[ U = aQ_1^2 + bQ_1Q_2 + cQ_2^2 \]

- Generalized coordinate $q_2$ when only $Q_1$ applies
  \[ \frac{\partial U}{\partial Q_2} \bigg|_{Q_2=0} = bQ_1 \]

- Generalized coordinate $q_1$ when only $Q_2$ applies
  \[ \frac{\partial U}{\partial Q_1} \bigg|_{Q_1=0} = bQ_2 \]

- Instead of measuring rotation due to force, it may be more convenient to measure displacement due to moment
Simple example

- What is the rotation of a cantilever beam under end force?
  - Requires dealing with bending moment that is function of location
    \[
    EI \frac{d^2w}{dx^2} = P(L - x) \quad w(0) = w'(0) = 0
    \]
    \[
    \frac{dw}{dx} = \frac{P}{EI} (Lx - x^2 / 2) \quad w'(L) = \frac{PL^2}{2EI}
    \]

- Instead calculate displacement at tip due to end moment (set M=P)
  \[
  EI \frac{d^2w}{dx^2} = M \quad w(0) = w'(0) = 0
  \]
  \[
  w = \frac{M}{2EI} x^2 \quad w(L) = \frac{ML^2}{2EI}
  \]

Beware of units!
Reading assignment

Section 5.5: What is more difficult about applying Castigliano’s theorem to statically indeterminate structures compared to statically determinate ones?