Prandtl stress function (Ludwig Prandtl 1875–1953)

- A trick to reduce three unknown stresses to a single unknown stress function
  
  Let: \( \sigma_{xz} = \frac{\partial \phi}{\partial y} \quad \sigma_{yz} = -\frac{\partial \phi}{\partial x} \)

- Solves exactly differential equations of equilibrium
  
  \[
  \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial x} = 0
  \]

- Substitute into compatibility equation
  
  \[
  \frac{\partial \tau_{zx}}{\partial y} - \frac{\partial \tau_{zy}}{\partial x} = -2G\theta \\
  \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\theta
  \]

- Mission accomplished!
Boundary conditions and moments

• Zero boundary stress

\[ \sigma_{zx} = \tau \sin \alpha \quad \sigma_{zy} = \tau \cos \alpha \]

\[ \sin \alpha = \frac{dx}{ds} \quad \cos \alpha = \frac{dy}{ds} \]

\[ \sigma_{zx} \cos \alpha - \sigma_{zy} \sin \alpha = 0 \Rightarrow \sigma_{zx} \frac{dy}{ds} - \sigma_{zy} \frac{dx}{ds} = 0 \]

\[ \frac{\partial \phi}{\partial y} \frac{dy}{ds} + \frac{\partial \phi}{\partial x} \frac{dx}{ds} = \frac{d\phi}{ds} = 0 \rightarrow \phi \text{ is constant on the boundary} \]

• Torque

\[ \sum M_z = T = \int_{A} (x \sigma_{zy} - y \sigma_{zx}) \, dx \, dy \Rightarrow \]

\[ T = -\int_{A} \left( x \frac{\partial \phi}{\partial x} + y \frac{\partial \phi}{\partial y} \right) \, dx \, dy = 2 \iint A \phi \, dx \, dy \]
Elliptical Cross Section

- For elliptical cross section
  \[
  \phi = B \left( \frac{x^2}{h^2} + \frac{y^2}{b^2} - 1 \right)
  \]

- Substituting into Poisson’s equation get
  \[
  B = -\frac{h^2 b^2 G}{h^2 + b^2}
  \]
Maximum stress and angle

• Stresses

\[ \sigma_{zx} = \frac{\partial \phi}{\partial x} = \frac{2By}{b^2} = -\frac{2h^2G\theta y}{h^2 + b^2} \]

\[ \sigma_{zy} = -\frac{\partial \phi}{\partial x} = -\frac{2Bx}{h^2} = \frac{2b^2G\theta x}{h^2 + b^2} \]

\[ \tau_{\text{max}} = \sigma_{zy(x=h)} = \frac{2b^2hG\theta}{h^2 + b^2} \]

• Torque

\[ T = \frac{2B}{h^2} \int x^2 dA + \frac{2B}{b^2} \int y^2 dA - 2B \int dA = \frac{2B}{h^2} I_y + \frac{2B}{b^2} I_x - 2BA \]

\[ T = -\pi Bhb \]
Torsional constant and polar moment of inertia

- Altogether get

$$
\tau_{\text{max}} = \frac{2T}{\pi bh^2}, \quad \theta = \frac{T(b^2 + h^2)}{G\pi b^3 h^3} = \frac{T}{G"J"}
$$

$$
G \frac{\pi b^3 h^3}{(b^2 + h^2)} = G"J"
$$

- How can we see that torsional constant is not equal to polar moment of inertia? Note that

$$
I_x = \frac{\pi bh^3}{4} \quad I_y = \frac{\pi bh^3}{4}
$$
Example

- Find how much we increase twist angle and maximum shear stress by using an elliptical cross section with b/h=2 instead of a circular section of same area. What is the error if we use J, not “J”

- Area Equality:
  \[ \pi bh = 2 \pi h^2 = \pi r^2 \]
  \[ \Rightarrow h = \frac{r}{\sqrt{2}}; b = r\sqrt{2} \]

- Maximum Stress:
  \[ \tau_{\text{max}} = \frac{2T}{\pi bh^2} = \frac{2T\sqrt{2}}{\pi r^3} \quad (41\% \text{ higher}) \]

- Angle of twist:
  \[ \theta = \frac{T(b^2 + h^2)}{G\pi b^3 h^3} = \frac{2.5Tr^2}{\pi r^6} \quad (25\% \text{ higher}) \]
  With J instead of "J"
  \[ \theta = \frac{4T}{G\pi bh(b^2 + h^2)} = \frac{1.6T}{\pi r^4} \quad (20\% \text{ lower}) \]
**Rectangle**: Solution difficult to obtain by Prandtl Stress function

\[
\tau_{\text{max}} = \frac{T}{k_2 (2b)(2h)^2}
\]

\[
\theta = \frac{T}{k_1 G (2b)(2h)^3}
\]

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The Prandtl soap-film analogy

- Textbook shows that for elastic membrane (e.g. soap film)

\[
\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -\frac{p}{S}
\]

- Why is that helpful?
Reading assignment

Sections 6.5-6: Using Table 6.1, for what aspect ratio is it reasonable to use the approximation of long and narrow cross section?

Source: www.library.veryhelpful.co.uk/Page11.htm