6.5 narrow rectangular cross section

- If section is long and narrow, soap bubble will be almost independent of long coordinate.
The Prandtl soap-film analogy

- Soap bubble equation

\[ \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = - \frac{p}{S} \]

- By comparing to the torsion equation get

\[ \frac{z}{p / S} = \frac{\phi}{2G\theta}, \quad \phi = \frac{2G\theta S}{p} z \]
Taking advantage of analogy

• Soap film deflection

\[ z = z_0 \left[ 1 - \left( \frac{y}{h} \right)^2 \right] \]

\[ \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -\frac{2z_0}{h^2} \]

• Get \( z_0 = \rho h^2 / 2S \) and

\[ \phi = G \theta h^2 \left[ 1 - \left( \frac{y}{h} \right)^2 \right] \]
Stresses and torsional constant

• Differentiate for stresses

\[ \sigma_{zx} = \frac{\partial \phi}{\partial y} = -2G\theta_y, \quad \sigma_{zy} = -\frac{\partial \phi}{\partial x} = 0 \quad \tau_{\text{max}} = 2G\theta h, \quad \text{for } y = \pm h \]

• Integrate for torque

\[ T = 2\int_{-b-h}^{b} \int_{-h}^{h} \phi \, dx \, dy = \frac{1}{3} G\theta(2b)(2h)^3 = G''J''\theta \quad "J'' = \frac{1}{3} (2b)(2h)^3 \]

• When section gets narrower for constant area, how do \( J \) and "\( J \)" change?

• Altogether

\[ \tau_{\text{max}} = \frac{3T}{(2b)(2h)^2} = \frac{2Th}{"J''}, \quad \theta = \frac{3T}{G(2b)(2h)^3} = \frac{T}{G''J''} \]
Cross sections made up of multiple narrow rectangles

- You just straighten them up!

\[
"J" = C \frac{1}{3} \sum_{i=1}^{n} (2b_i)(2h_i)^3
\]

\[
\tau_{\text{max}} = \frac{2T h_{\text{max}}}{"J"}, \quad \theta = \frac{T}{G"J"}
\]

- If we have a rectangle, would we gain by halving the thickness on half the length and increasing it by 50% on the other half?
Example

- For the section shown calculate the maximum shear stress and the angle of twist per unit length when the member is subjected to torque $T = 100 \text{ in.lb. } G=12\text{Mpsi}$

Solution : Total length of the member, $2b = 2 \times (1.125 - 0.05) + 1.5$

$2b = 3.65 \text{ in} \quad \text{and} \quad 2h = 0.05 \text{ in}$

$b/h = 73 > 10 \implies \text{rectangular approximation used.}$

\[
\tau_{\text{max}} = \frac{3T}{(2b)(2h)^2} = \frac{3(100)}{(3.65)(0.05)^2} = 32,876 \text{ ksi}
\]

\[
\theta = \frac{3T}{G(2b)(2h)^3} = \frac{3(100)}{12 \times 10^6 (3.65)(0.05)^3} = 0.0548 \text{ rad/in}
\]
Torsion of rectangular cross section members

- Geometry:

\[ \phi = G \theta \left( h^2 - x^2 \right) - \frac{32 G \theta h^2}{\pi^3} \sum_{n=1,3,5,...}^{\infty} \frac{(-1)^{n-1}}{n^3 \cosh \frac{n \pi b}{2h}} \cos \frac{n \pi x}{2h} \cosh \frac{n \pi y}{2h} \]

- How fast does this converge?
Bringing to nondimensional form

- **Stiffness and stress**

\[ "J" = k_1 (2h)^3 (2b) \]

\[ k_1 = \frac{1}{3} \left[ 1 - \frac{192}{\pi^5} \left( \frac{h}{b} \right) \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^5} \tanh \frac{n\pi b}{2h} \right] \]

\[ \tau_{\text{max}} = 2G\theta h k_2 / k_1 \quad \frac{k_2}{k_1} = 1 - \frac{8}{\pi^2} \sum_{n=1,3,5,\ldots} \left( \frac{1}{n^2 \cosh \frac{n\pi b}{2h}} \right) \]

### Torsional Parameters for Rectangular Cross Sections

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<th>1</th>
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<th>2</th>
<th>2.5</th>
<th>3</th>
<th>4</th>
<th>6</th>
<th>10</th>
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<tr>
<td>(k_1)</td>
<td>0.141</td>
<td>0.196</td>
<td>0.229</td>
<td>0.249</td>
<td>0.263</td>
<td>0.281</td>
<td>0.299</td>
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<tr>
<td>(k_2)</td>
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<td>0.246</td>
<td>0.256</td>
<td>0.267</td>
<td>0.282</td>
<td>0.299</td>
<td>0.312</td>
<td>0.333</td>
</tr>
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Hollow thin-wall torsion members and multiply connected cross sections

- Hollow sections much more efficient than open ones
- Compare a hollow cylinder of radius \( R \) and thickness \( t \) to same cross section with a slit cut open

\[
J_{\text{cut}} = \frac{1}{3} (2\pi r)t^3 \quad J_{\text{pipe}} = 2\pi r^3 t
\]

\[
\frac{J_{\text{pipe}}}{J_{\text{cut}}} = 3 \left( \frac{r}{t} \right)^2
\]
Reading assignment

Section 6.7: Question: For a hollow cross section do we gain by reducing the thickness on half the perimeter and increasing it on the other half? Why?