Problem 1

Verify by substitution that each given function is a solution of the given differential equation. Primes denote derivatives with respect to x.

\[ y' + 2y = 0 \quad ; \quad y = 3e^{-2x} \]

Eq.1: \( y' + 2y = 0 \)  
Eq.2: \( y = 3e^{-2x} \)

Differentiating Eq.2 with respect to x yields:

a) \( y' = -6e^{-2x} \)

From Eq.2,

b) \( 2y = 6e^{-2x} \)

Substituting a and b into Eq.1 yields:

\[-6e^{-2x} + 6e^{-2x} = 0\]
Problem 6

Verify by substitution that each given function is a solution of the given differential equation. Primes denote derivatives with respect to x.

\[ y'' + 4y' + 4y = 0; \quad y_1 = e^{-2x}, y_2 = xe^{-2x} \]

The problem will first be verified for \( y_1 \), and then the problem will be verified again for \( y_2 \).

With \( y_1 \):

\[ \text{Eq.1: } y'' + 4y' + 4y = 0 \quad \text{Eq.2: } y_1 = e^{-2x} \]

Differentiating Eq.2 with respect to x and multiplying by 4 yields:

a) \( 4y_1' = -8e^{-2x} \)

Differentiating again with respect to x yields:

b) \( y_1'' = 4e^{-2x} \)

Multiplying Eq.2 by 4 yields:

c) \( 4y_1 = 4e^{-2x} \)

Substituting a, b, and c into Eq.1 yields:

\[ 4e^{-2x} - 8e^{-2x} + 4e^{-2x} = 0 \]

\[ 0 = 0 \]

Now the problem will be verified for \( y_2 \).

\[ \text{Eq.1: } y'' + 4y' + 4y = 0 \quad \text{Eq.2: } y_2 = xe^{-2x} \]

Differentiating Eq.2 with respect to x and multiplying by 4 yields:

a) \( 4y_2' = 4e^{-2x} - 8xe^{-2x} \)

Differentiating with respect to x again yields:

b) \( y_2'' = -2e^{-2x} - 2e^{-2x} + 4xe^{-2x} = 4xe^{-2x} - 4e^{-2x} \)

Multiplying Eq.2 by 4 yields:

c) \( 4y_2 = 4xe^{-2x} \)
Substituting $a$, $b$, and $c$ into Eq. 1 yields:

$$4xe^{-2x} - 4e^{-2x} + 4e^{-2x} - 8xe^{-2x} + 4xe^{-2x} = 0$$

$$0 = 0$$
Problem 15

Substitute \( y = e^{rx} \) into the given differential equation to determine all values of the constant \( r \) for which \( y = e^{rx} \) is a solution of the equation.

\[ y'' + y' - 2y = 0 ; \quad y = e^{rx} \]

Eq.1: \( y'' + y' - 2y = 0 \) \quad Eq.2: \( y = e^{rx} \)

Differentiating Eq.2 with respect to \( x \) yields:

a) \( y' = re^{rx} \)

Differentiating with respect to \( x \) again yields:

b) \( y'' = r^2 e^{rx} \)

Multiplying Eq.2 by \(-2\) yields:

c) \(-2y = -2e^{rx}\)

Substituting a, b, and c into Eq.1 yields:
\[ r^2 e^{rx} + re^{rx} - 2e^{rx} = 0 \]

Dividing both sides by \( e^{rx} \) gives:
\[ r^2 + r - 2 = 0 \]

The above quadratic equation is solved as follows:
\[
r = \frac{-1 \pm \sqrt{1 - 4(1)(-2)}}{2(1)} = \frac{-1 \pm \sqrt{9}}{2} = \frac{-1 \pm 3}{2}
\]

\[
r = -2, 1
\]
**Problem 22**

First verify that \( y(x) \) satisfies the given differential equation. Then determine a value of the constant \( C \) so that \( y(x) \) satisfies the given initial condition.

\[ e^y y' = 1; \ y(x) = \ln(x + C), \ y(0) = 0 \]

Eq.1: \( e^y y' = 1 \)

Substituting the given function \( y(x) = \ln(x + C) \) into Eq.1 gives:

\[ e^y = e^{\ln(x+c)} = x + c \]

and

\[ y' = \frac{1}{x + c} \]

Eq.1 becomes:

\[ \frac{x + c}{x + c} = 1 \quad \therefore \]

Now \( C \) must be determined using the initial condition \( y(0) = 0 \).

\[ y(0) = \ln(0 + C) = 0 \]

\[ \ln(C) = 0 \]

\[ \therefore C = 1 \]
Problem 33

Write – in the manner of Eqs.(3) through (6) of this section – a differential equation that is a mathematical model of the situation described.

The time rate of change of the velocity $v$ of a coasting motorboat is proportional to the square of $v$.

$$\frac{dv}{dt} \equiv \text{time rate of change of } v$$

$k \equiv \text{proportionality constant}$

$$\frac{dv}{dt} = kv^2$$