Problem 7

Find general solutions (implicit if necessary, explicit if convenient) of the differential equations.

\[
\frac{dy}{dx} = (64xy)^{1/3}
\]

Separating the variables yields:

\[
y^{-1/3} \, dy = 64^{1/3} \, x^{1/3} \, dx
\]
\[
y^{-1/3} \, dy = 4x^{1/3} \, dx
\]

Integrating both sides to obtain \( y(x) \) yields:

\[
\int y^{-1/3} \, dy = \int 4x^{1/3} \, dx
\]
\[
\frac{3y^{2/3}}{2} = 3x^{4/3} + C
\]
\[
y^{2/3} = 2x^{4/3} + \frac{2}{3}C
\]

\[
y(x) = (2x^{4/3} + \tilde{C})^{3/2}
\]

Where \( \tilde{C} = \frac{2}{3}C \)
Problem 9

Find general solutions (implicit if necessary, explicit if convenient) of the differential equations.

\((1 - x^2) \frac{dy}{dx} = 2y\)

Separating the variables yields:
\[
\frac{dy}{y} = \frac{2dx}{1-x^2}
\]

Integrating both sides to obtain \(y(x)\) yields:
\[
\int \frac{dy}{y} = \int \frac{2dx}{1-x^2}
\]
\[
\frac{2}{1-x^2} \text{ can be decomposed into } \frac{1}{1+x} + \frac{1}{1-x}
\]

The above integral becomes
\[
\int \frac{dy}{y} = \int \frac{dx}{1+x} + \frac{dx}{1-x}
\]
\[
\ln y = \ln(1+x) - \ln(1-x) + C
\]
\[
y = e^{\ln(1+x) - \ln(1-x) + C}
\]
\[
y = e^{\ln(1+x)} e^{-\ln(1-x)} e^C
\]
\[
y(x) = \tilde{C} \frac{1+x}{1-x}
\]

Where \(\tilde{C} = e^C\)
Problem 25

Find explicit particular solutions of the initial value problem.

\[ x \frac{dy}{dx} - y = 2x^2 y \quad , \quad y(1) = 1 \]

The variables in the differential equation are separated as follows.

\[ x \frac{dy}{dx} = y(2x^2 + 1) \]

\[ \frac{dy}{y} = (2x^2 + 1) \frac{dx}{x} = (2x + \frac{1}{x}) dx \]

Integrating both sides to obtain \( y(x) \) yields:

\[ \int \frac{dy}{y} = \int (2x + \frac{1}{x}) dx \]

\[ \ln y = x^2 + \ln x + C \]

\[ y = e^{x^2 + \ln x + C} \]

\[ y = \tilde{C} e^{x^2} \]

Where \( \tilde{C} = e^C \)

The initial condition is used to solve for the constant.

\[ y(1) = \tilde{C} e^1 (1) = 1 \]

\[ \therefore \]

\[ \tilde{C} = \frac{1}{e} \]

\[ y(x) \text{ becomes } y(x) = \frac{xe^{x^2}}{e} \]

Simplifying gives

\[ y(x) = xe^{(x^2 - 1)} \]
Problem 29

(Population growth) A certain city had a population of 25000 in 1960 and a population of 30000 in 1970. Assume that its population will continue to grow exponentially at a constant rate. What population can its city planners expect in the year 2000?

From pg.36, \( P(t) = P_0 e^{kt} \), where \( t \) is the time in years.

Assuming 1960 is the initial time, the given information is as follows:
\[
P_0 = 25000 \\
P(10) = 30000
\]

\( k \) is solved for using the given information as follows:
\[
P(10) = 30000 = 25000 e^{k \cdot 10} \\
\frac{6}{5} = e^{k \cdot 10} \\
k = \frac{\ln\left(\frac{6}{5}\right)}{10}
\]

Now the population in the year 2000 or, \( P(40) \), can be found.
\[
P(40) = 25000 e^{\frac{\ln\left(\frac{6}{5}\right)}{10}} = 25000 \left(\frac{6}{5}\right)^4
\]

\[P(40) = 51,840 \text{ people}\]
Problem 31

(Radiocarbon dating) Carbon extracted from an ancient skull contained only one-sixth as much $^{14}$C as carbon extracted from present-day bone. How old is the skull?

The governing differential equation is given on pg.35

$$\frac{dN}{dt} = -kN$$

The solution of the differential equation is

$$N(t) = N_0 e^{-kt}$$

It is given that at the current time, $N = \frac{N_0}{6}$.

From pg.35, $k = 0.0001216$ if $t$ is in years.

The equation to be solved is:

$$\frac{N_0}{6} = \frac{N_0}{6} e^{-(0.0001216)t}$$

Dividing both sides by $N_0$ gives

$$\frac{1}{6} = e^{-(0.0001216)t}$$

$$\ln(\frac{1}{6}) = -0.0001216t$$

$$t = 14,735 \text{ years}$$
Problem 39

A pitcher of buttermilk initially at 25° C is to be cooled by setting it on the front porch, where the temperature is 0° C. Suppose that the temperature of the buttermilk has dropped to 15° C after 20 min. When will it be at 5° C.

The governing differential equation is Eq.19 pg.37.
\[
\frac{dT}{dt} = -k(A - T)
\]

\(A\) is the medium temperature and time is measured in minutes for this problem. It is given that \(A\) is 0° C, so the differential equation becomes
\[
\frac{dT}{dt} = -kT
\]

Separating variables yields:
\[
\frac{dT}{T} = -kdt
\]

Integrating both sides to get \(T(t)\) yields:
\[
\int \frac{dT}{T} = \int -kdt
\]

\[\ln T = -kt + C\]

\[T = e^{-kt+C}\]

\[T = \tilde{C}e^{-kt}\]

Where \(\tilde{C} = e^C\)

The initial condition is given as \(T(0) = 25\). Substituting the initial condition into the above equation gives
\[T(0) = 25 = \tilde{C}(1)\]
\[\tilde{C} = 25\]

\[\therefore\]
\[T = 25e^{-kt}\]

It is also given that \(T(20) = 15\). This information is used to solve for \(k\).
\[T(20) = 15 = 25e^{-k(20)}\]
\[k = -\frac{\ln(\frac{15}{25})}{20}\]

The time when the temperature reaches 5° C can be solved for as follows:
\[ T(t) = 5 = 25e^{\frac{\ln(\frac{1}{3})}{20}t} \]

\[ \frac{1}{5} = e^{\frac{\ln(\frac{1}{3})}{20}} \]

\[ \ln(\frac{1}{3}) = \frac{\ln(\frac{1}{3})}{20}t \]

\[ t = 63 \text{ min} \]
Problem 44

According to one cosmological theory, there were equal amounts of the two uranium isotopes \(^{235}\text{U}\) and \(^{238}\text{U}\) at the creation of the universe in the “big bang.” At present there are 137.7 atoms of \(^{238}\text{U}\) for each atom of \(^{235}\text{U}\). Using the half-lives \(4.51 \times 10^9\) years for \(^{238}\text{U}\) and \(7.10 \times 10^8\) years for \(^{235}\text{U}\), calculate the age of the universe.

M will denote the number of atoms of \(^{235}\text{U}\) and N will denote the number of atoms of \(^{238}\text{U}\). The form of the governing equation is

\[ N = N_0 e^{-kt} \]

It can be shown that the half-life, \(\tau\), is

\[ \tau = -\frac{\ln 2}{k} \]

(see pg.37)

The constant, k, can be solved for because the half-lives are given.

\[ k = -\frac{\ln 2}{\tau} \]

\[ k_{238} = 1.54 \times 10^{-10} \]

\[ k_{235} = 9.76 \times 10^{-10} \]

\[ N = N_0 e^{-1.54 \times 10^{-10} t} \]

\[ M = M_0 e^{-9.76 \times 10^{-10} t} \]

The initial time is the beginning of the universe. Dividing both equations at the present time gives

\[ \frac{N}{M} = 137.7 = e^{-t(1.54 \times 10^{-10} - 9.76 \times 10^{-10})} \]

\[ \ln 137.7 = -t(1.54 \times 10^{-10} - 9.76 \times 10^{-10}) \]

\[ t = 5.99 \text{ billion years} \]
Problem 47

A certain piece of dubious information about phenylethylamine in the drinking water began to spread one day in a city with a population of 100,000. Within a week, 10000 people had heard this rumor. Assume that the rate of increase of the number who have heard the rumor is proportional to the number who have not yet heard it. How long will it be until half the population of the city has heard the rumor?

An ODE must be formed to fit the above information. Let \( N \) be the number of people, in thousands, who have heard the rumor.

The ODE is:

\[
\frac{dN}{dt} = k(100 - N)
\]

The above ODE states that the rate of increase of the number who have heard the rumor is proportional to the number who have not yet heard it, as stated in the problem statement. Time is measured in days. “100-N” is the total population minus the people who have heard the rumor, and thus the number of people who have not heard the rumor.

Separating variables yields:

\[
\frac{dN}{100 - N} = kdt
\]

Integrating both sides to obtain \( N(t) \) yields:

\[
\int \frac{dN}{100 - N} = \int kdt
\]

\[-\ln(100 - N) = kt + C\]

\[100 - N = e^{-(kt+C)}\]

\[100 - N = \tilde{C}e^{-kt}\]

Where \( \tilde{C} = e^C \)

Initially, no one has heard the rumor, so \( N(0) = 0 \). This initial condition is used to solve for \( \tilde{C} \).

\[100 - 0 = \tilde{C}(1)\]

\[\tilde{C} = 100\]

\( N(7) = 10 \) is given and is used to solve for \( k \).

\[N(7) = 10 = 100 - 100e^{-k(7)}\]

\[90 = 100e^{-k7}\]

\[k = -\frac{\ln(\frac{9}{10})}{7}\]
Now the time when half the population has heard the rumor can be found.

\[
50 = 100 - 100e^{-\frac{\ln\left(\frac{9}{10}\right)}{7}t}
\]

\[
50 = 100e^{-\frac{\ln\left(\frac{9}{10}\right)}{7}t}
\]

\[
\frac{1}{2} = e^{-\frac{\ln\left(\frac{9}{10}\right)}{7}t}
\]

\[
\ln\left(\frac{1}{2}\right) = -\frac{\ln\left(\frac{9}{10}\right)}{7}t
\]

Solving for \(t\) gives:

\[
t = 46 \text{ days}
\]