Solutions 1.7-Page 79

Problem 3

Separate the variables and use partial fractions to solve the initial value problems.

\[
\frac{dx}{dt} = 4x(7-x) \quad , \quad x(0) = 11
\]

Separating variables gives \( \frac{dx}{4x(7-x)} = dt \). Factoring using partial fractions yields:

\[
dx \left( \frac{1}{28x} + \frac{1}{28(7-x)} \right) = dt
\]

\[
dx \left( \frac{1}{x} + \frac{1}{7-x} \right) = 28dt
\]

Integrating both sides will give \( x(t) \).

\[
\int dx \left( \frac{1}{x} + \frac{1}{7-x} \right) = \int 28dt
\]

\[
\ln x - \ln(7-x) = 28t + C
\]

\[
\ln \left( \frac{x}{7-x} \right) = 28t + C
\]

\[
e^{\ln \left( \frac{x}{7-x} \right)} = e^{28t+C}
\]

\[
\frac{x}{7-x} = \tilde{C}e^{28t}
\]

Note: \( \tilde{C} = e^C \)

The initial condition is used to find \( \tilde{C} \).
\[ x(0) = 11 = -\frac{11}{4} = \tilde{C}(1) \]

\[ \tilde{C} = -\frac{11}{4} \]

\[ : \]

\[ \frac{x}{7-x} = -\frac{11}{4} e^{28t} \]

\[ or \]

\[ \frac{7-x}{x} = -\frac{4}{11} e^{-28t} \]

\[ \frac{7}{x} - 1 = -\frac{4}{11} e^{-28t} \]

\[ \frac{7}{x} = 1 - \frac{4}{11} e^{-28t} \]

\[ \frac{7}{x} = \frac{11}{11} - \frac{4}{11} e^{-28t} \]

Rearranging and solving for \( x \) yields:

\[ x(t) = \frac{77}{11 - 4e^{-28t}} \]
Problem 5

The time rate of change of a rabbit population $P$ is proportional to the square root of $P$. At time $t = 0$ (months) the population numbers 100 rabbits and is increasing at the rate of 20 rabbits per month. How many rabbits will there be one year later?

The given information translates into equation form as follows:

$$\frac{dP}{dt} = k\sqrt{P}$$

The initial conditions are used to find $k$.

$$\frac{dP}{dt} = 20 = k\sqrt{100}$$
$$\therefore k = 2$$

Separating variables and integrating will give $P(t)$.

$$\int \frac{dP}{\sqrt{P}} = \int 2dt$$
$$2P^{1/2} = 2t + C$$
$$P = \left( \frac{2t + C}{2} \right)^2$$

$P(0) = 100 = \left( \frac{C}{2} \right)^2$
$$\therefore C = 20$$
$$P(t) = (t + 10)^2$$

$P(12) = 484$
Problem 7

Suppose that when a certain lake is stocked with fish, the birth and death rates $\beta$ and $\delta$ are both inversely proportional to $\sqrt{P}$. (a) Show that

$$P(t) = \left(\frac{1}{2}kt + \sqrt{P_0}\right)^2$$

where k is a constant. (b) If $P_0 = 100$ and after 6 months there are 169 fish in the lake, how many will there be after 1 year?

(a) The governing differential equation is $\frac{dP}{dt} = (\beta - \delta)P$ (see pg70-71). The statement that “$\beta$ and $\delta$ are both inversely proportional to $\sqrt{P}$” translates into equation form as follows:

$$\beta = \frac{A}{\sqrt{P}} : \delta = \frac{B}{\sqrt{P}} \quad \text{where A and B are arbitrary constants.}$$

The differential equation now becomes:

$$\frac{dP}{dt} = \left(\frac{A}{\sqrt{P}} - \frac{B}{\sqrt{P}}\right)P$$

$$\frac{dP}{dt} = \frac{1}{\sqrt{P}}(A - B)P$$

$$\frac{dP}{dt} = k\sqrt{P}$$

where $k = A - B$.

Separating variables and integrating both sides gives:

\[ \int P^{-1/2}dP = \int kdt \]

$$2P^{1/2} = kt + C$$

$$P = \left(\frac{kt}{2} + \tilde{C}\right)^2$$

$\tilde{C}$ is $\sqrt{P_0}$ because:

$$P(0) = P_0 = \tilde{C}^2$$

Therefore

$$P(t) = \left(\frac{kt}{2} + \sqrt{P_0}\right)^2$$

(b) The given information is used to find $k$ (with $t$ in months).
\[
P(6) = 169 = \left( \frac{k(6)}{2} + \sqrt{100} \right)^2
\]

\[
\therefore \quad k = 1
\]

and

\[
P(12) = \left( \frac{1(12)}{2} + \sqrt{100} \right)^2 = 256
\]

\[
P(12) = 256
\]
Problem 11

Consider a population $P(T)$ satisfying the logistic equation $\frac{dP}{dt} = aP - bP^2$, where $B = aP$ is the time rate at which births occur and $D = bP^2$ is the rate at which deaths occur. If the initial population is $P(0) = P_0$, and $B_0$ births per month and $D_0$ deaths per month are occurring at time $t = 0$, show that the limiting population is $M = \frac{B_0P_0}{D_0}$.

The differential equation can be rewritten as follows:

$$\frac{dP}{dt} = P(a - bP^2)$$
$$\frac{dP}{dt} = bP\left(\frac{a}{b} - P\right)$$

Recall that the logistic equation can be written as $\frac{dP}{dt} = k(P - M)$. Comparing this form to the above differential equation implies that $M = \frac{a}{b}$. From the given information, $a$ and $b$ at $t = 0$ can be found and substituted into the equation for $M$.

$$a_0 = \frac{B_0}{P_0}$$
$$b_0 = \frac{D_0}{P_0^2}$$

Substitution yields:

$$M = \frac{B_0P_0}{D_0}$$
Problem 13

Consider a rabbit population $P(t)$ satisfying the logistic equation as in Problem 11. If the initial population is 240 rabbits and there are 9 births per month and 12 deaths per month occurring at time $t = 0$, how many months does it take for $P(t)$ to reach 105% of the limiting population $M$?

From the relations in Problem 11, $b = k = D_0 / P_0^2 = 12 / 240^2 = 1 / 4800$. The details for finding $P(t)$ from the logistic equation can be found on pg.72. The result is:

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-kt}}$$

From Problem 11, $M = \frac{B_0 P_0}{D_0} = \frac{(9)(240)}{12} = 180$. Substituting the known values into the above equation gives

$$P(t) = \frac{180(240)}{240 + (180 - 240)e^{-180t/4800}} = \frac{43200}{240 - 60e^{-3t/80}}.$$ The time when $P(t)$ is 105% of $M$ can now be found.

$$P(t) = 1.05(180) = \frac{43200}{240 - 60e^{-3t/80}}$$

Solving for $t$ (in months) gives

$$t = 44.22 \text{ months}$$