Problem 1

Find the general solutions in powers of $x$ of the differential equations. State the recurrence relations and the guaranteed radius of convergence in each case.

$$(x^2 - 1)y'' + 4xy' + 2y = 0$$

Substituting $y = \sum_{n=0}^{\infty} c_n x^n$, $y' = \sum_{n=1}^{\infty} nc_n x^{n-1}$, and $y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$ into the differential equation yields

$$(x^2 - 1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + 4x\sum_{n=1}^{\infty} nc_n x^{n-1} + 2\sum_{n=0}^{\infty} c_n x^n = 0$$

Simplifying further,

$$\sum_{n=2}^{\infty} n(n-1)c_n x^n - \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + 4\sum_{n=1}^{\infty} nc_n x^n + 2\sum_{n=0}^{\infty} c_n x^n = 0$$

The first and third summations can start at $n = 0$ and no additional nonzero terms will be added. However, the second summation must be rewritten to start at $n = 0$ such that the identity principle can be used.

$$\sum_{n=0}^{\infty} n(n-1)c_n x^n - \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n + 4\sum_{n=0}^{\infty} nc_n x^n + 2\sum_{n=0}^{\infty} c_n x^n = 0$$

Using the identity principle and summing coefficients yields

$$n(n-1)c_n - (n+2)(n+1)c_{n+2} + 4nc_n + 2c_n = 0$$

$$c_{n+2} = \frac{n(n-1)c_n + 4nc_n + 2c_n}{(n+2)(n+1)} = \frac{c_n[n(n-1) + 4n + 2]}{(n+2)(n+1)} = \frac{c_n[n^2 + 3n + 2]}{(n+2)(n+1)} = \frac{c_n(n+2)(n+1)}{(n+2)(n+1)}$$

So the recurrence formula is

$$c_{n+2} = c_n$$

From this recurrence formula, it is evident that coefficients with even indices $(n = 0, 2, 4, \ldots)$ are equal to $c_0$, and coefficients with odd indices $(n = 1, 3, 5, \ldots)$ are equal to $c_1$.  

Therefore \[ y = \sum_{n=0}^{\infty} c_n x^n = c_0 \sum_{n=0}^{\infty} x^{2n} + c_1 \sum_{n=0}^{\infty} x^{2n+1} \]

Expanding the summations gives

\[ y = c_0 \sum_{n=0}^{\infty} x^{2n} + c_1 \sum_{n=0}^{\infty} x^{2n+1} \]
\[ = c_0 \left[ 1 + x^2 + x^4 + \ldots \right] + c_1 \left[ x + x^3 + x^5 + \ldots \right] \]
\[ = c_0 \left[ 1 + x^2 + x^4 + \ldots \right] + c_1 x \left[ 1 + x^2 + x^4 + \ldots \right] \]

The first summations match the form of \[ \frac{1}{1-x} \] if the variable is \( x^2 \) instead of \( x \).

So \[ y = \frac{c_0}{1-x^2} + \frac{c_1 x}{1-x^2} \]

The notation of the text is to write a differential equation as \[ A(x)y'' + B(x)y' + C(x)y = 0 \]. For this problem, \( A(x) = x^2 - 1 \), which has a singular point at \( x = 1, -1 \). The distance from one of these points to \( a = 0 \) is 1, so the guaranteed radius of convergence is at least 1.

\[ \rho \geq 1 \]
Problem 3

Find the general solutions in powers of x of the differential equations. State the recurrence relations and the guaranteed radius of convergence in each case.

\[ y'' + xy' + y = 0 \]

Substituting \( y = \sum_{n=0}^{\infty} c_n x^n \), \( y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \), and \( y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} \) into the differential equation yields

\[ \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x \sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0 \]

Simplifying further,

\[ \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0 \]

The second summation can start at \( n = 0 \) and no additional nonzero terms will be added. The first summation must be rewritten to start at \( n = 0 \) such that the identity principle can be used.

\[ \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n + \sum_{n=0}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0 \]

Using the identity principle and summing coefficients yields

\[ (n + 2)(n + 1)c_{n+2} + nc_n + c_n = 0 \]

\[ c_{n+2} = \frac{nc_n + c_n}{(n + 2)(n + 1)} = \frac{-c_n + \frac{c_n(n+1)}{n+2}}{(n + 2)(n + 1)} \]

The recurrence formula is

\[ c_{n+2} = -\frac{c_n}{n+2} \]

The coefficients can be written in terms of \( c_0 \) and \( c_1 \) as follows:
For even indices, \[ c_{2n} = \frac{(-1)^n c_0}{n!2^n} \]

For odd indices, \[ c_{2n+1} = \frac{(-1)^n c_1}{(2n+1)!!} \]

Therefore, \[
y = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{(-1)^n c_0}{n!2^n} x^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n c_1}{(2n+1)!!} x^{2n+1}
\]

The notation of the text is to write a differential equation as \[ A(x)y^{(n)} + B(x)y' + C(x)y = 0. \] For this problem, \( A(x) = 1 \) which has no singular point.

\[ \rho = \infty \]
Problem 7

Find the general solutions in powers of x of the differential equations. State the recurrence relations and the guaranteed radius of convergence in each case.

\((x^2 + 3)y'' - 7xy' + 16y = 0\)

Substituting \(y = \sum_{n=0}^{\infty} c_n x^n\), \(y' = \sum_{n=1}^{\infty} nc_n x^{n-1}\), and \(y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}\) into the differential equation yields

\((x^2 + 3)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 7x \sum_{n=1}^{\infty} nc_n x^{n-1} + 16\sum_{n=0}^{\infty} c_n x^n = 0\)

Simplifying further,

\(\sum_{n=2}^{\infty} n(n-1)c_n x^n + 3 \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 7 \sum_{n=1}^{\infty} nc_n x^{n-1} + 16 \sum_{n=0}^{\infty} c_n x^n = 0\)

The first and third summations can start at \(n = 0\) and no additional nonzero terms will be added. However, the second summation must be rewritten to start at \(n = 0\) such that the identity principle can be used.

\(\sum_{n=0}^{\infty} n(n-1)c_n x^n + 3 \sum_{n=0}^{\infty} (n+2)(n+1)c_n x^n - 7 \sum_{n=0}^{\infty} nc_n x^n + 16 \sum_{n=0}^{\infty} c_n x^n = 0\)

Using the identity principle and summing coefficients yields

\(\sum_{n=0}^{\infty} n(n-1)c_n x^n + 3 \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n - 7 \sum_{n=0}^{\infty} nc_n x^n + 16 \sum_{n=0}^{\infty} c_n x^n = 0\)

\(n(n-1)c_n + 3(n+2)(n+1)c_{n+2} - 7nc_n + 16c_n = 0\)

\(c_{n+2} = \frac{7nc_n - n(n-1)c_n - 16c_n}{3(n+2)(n+1)} = \frac{c_n[7n - n(n-1) - 16]}{3(n+2)(n+1)} = \frac{c_n[7n - n(n-1) - 16]}{3(n+2)(n+1)}\)

\(c_{n+2} = \frac{c_n}{3(n+2)(n+1)}[-n^2 + 8n - 16]\)

The recurrence formula is

\[c_{n+2} = -\frac{c_n(n-4)^2}{3(n+2)(n+1)}\]
The coefficients can be written in terms of \( c_0 \) and \( c_1 \) as follows:

\[
\begin{align*}
  n = 0: & \quad c_2 = -\frac{16c_0}{6} = -\frac{8c_0}{3} \\
  n = 1: & \quad c_3 = -\frac{9c_1}{18} = -\frac{1c_1}{2} \\
  n = 2: & \quad c_4 = -\frac{4c_2}{36} = \frac{32c_0}{108} = \frac{8c_0}{27} \\
  n = 3: & \quad c_5 = -\frac{c_3}{60} = \frac{c_1}{120} \\
  n = 4: & \quad c_6 = 0 \\
  n = 5: & \quad c_7 = -\frac{c_5}{3(7)(6)} = \frac{c_1}{3(7)(6)(120)} = \frac{c_1}{7! \cdot 3} \\
  n = 7: & \quad c_9 = -\frac{9c_7}{3(9)(8)} = -\frac{c_1}{3(9)(8)(7!)(3)} = \frac{9c_1}{9! \cdot 3^2} = 9 \cdot \frac{(3!!)^2 c_1}{9! \cdot 3^4} \\
  n = 9: & \quad c_{11} = -\frac{25c_9}{3(11)(10)} = \frac{25(9)(9)c_1}{3(11)(10)(9!)(3^4)} = 9 \cdot \frac{(15!!)^2 c_1}{11! \cdot 3^5}
\end{align*}
\]

Starting at \( n = 3 \), the pattern that governs the odd index coefficients is

\[ c_{2n+1} = 9 \left[ \frac{(2n-5)!!}{(2n+1)!3^n} \right]^2 (-1)^n \quad \text{for } n \geq 3 \]

Therefore

\[
y = \sum_{n=0}^{\infty} c_n x^n = c_0 \left( 1 - \frac{8}{3} x^2 + \frac{8}{27} x^4 \right) + c_1 \left[ x - \frac{1}{2} x^3 + \frac{1}{120} x^5 + 9 \sum_{n=3}^{\infty} \frac{(2n-5)!!}{(2n+1)!3^n} x^{2n+1} \right]
\]

The notation of the text is to write a differential equation as

\[ A(x)y'' + B(x)y' + C(x)y = 0 \]. For this problem, \( A(x) = x^2 + 3 \), which has a singular point at \( x = \pm \sqrt{3} \). The distance from one of these points to \( a = 0 \) is \( \sqrt{3} \), so the guaranteed radius of convergence is at least \( \sqrt{3} \).

\[ \rho \geq \sqrt{3} \]
Problem 23

Find a three-term recurrence relation for solutions of the form $y = \sum_{n=0}^{\infty} c_n x^n$. Then find the first three nonzero terms in each of two linearly independent solutions.

$$y'' + (1 + x)y = 0$$

Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ and $y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$ into the differential equation yields

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + (1 + x) \sum_{n=0}^{\infty} c_n x^n = 0$$

Simplifying further,

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=2}^{\infty} c_n x^n + \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

The first summation can be started at $n = 0$ to get a $x^n$ term. The third summation must start at $n = 1$ in order to get a $x^n$ term.

$$\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n + \sum_{n=0}^{\infty} c_n x^n + \sum_{n=1}^{\infty} c_{n-1} x^n = 0$$

The common range is $n \geq 1$, so the terms corresponding to $n = 0$ must be brought out.

$$2c_2 + c_0 + \sum_{n=1}^{\infty} [(n+2)(n+1)c_{n+2} + c_n + c_{n-1}]x^n = 0$$

The identity principle implies that $2c_2 + c_0 = 0$ and $[(n+2)(n+1)c_{n+2} + c_n + c_{n-1}] = 0$.

The recurrence formula for $n \geq 1$ is $c_{n+2} = -\frac{c_n + c_{n-1}}{(n+2)(n+1)}$

The first linearly independent solution can be obtained by setting $c_0 = 1$ and $c_1 = 0$. Therefore $c_2 = -1/2$. The next coefficient is found using the recurrence formula.

$$n = 1: c_3 = -\frac{c_1 + c_0}{6} = -\frac{1 + 0}{6} = -1/6$$
The next linearly independent solution can be obtained by setting $c_0 = 0$ and $c_1 = 1$. Therefore $c_2 = 0$. The next coefficients are found using the recurrence formula.

\begin{align*}
n = 1: \quad c_3 &= -\frac{c_1 + c_0}{6} = -\frac{1 + 0}{6} = -\frac{1}{6} \\
n = 2: \quad c_4 &= -\frac{c_2 + c_1}{12} = -\frac{0 + 1}{6} = -\frac{1}{12}
\end{align*}

\[
y_2 = x - \frac{1}{6} x^3 - \frac{1}{12} x^4 + ...
\]
Problem 27

Solve the initial value problem

\[ y'' + xy' + (2x^2 + 1)y = 0; \quad y(0) = 1, \ y'(0) = -1 \]

Determine sufficiently many terms to compute \( y(1/2) \) accurate to four decimal places.

Substituting \( y = \sum_{n=0}^{\infty} c_n x^n \), \( y' = \sum_{n=1}^{\infty} nc_n x^{n-1} \), and \( y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} \) into the differential equation yields

\[ \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x \sum_{n=1}^{\infty} nc_n x^{n-1} + (2x^2 + 1) \sum_{n=0}^{\infty} c_n x^n = 0 \]

Simplifying further,

\[ \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} nc_n x^n + 2 \sum_{n=0}^{\infty} c_{n+2} x^{n+2} + \sum_{n=0}^{\infty} c_n x^n = 0 \]

All of the summations must be written in terms of \( x^n \) in order to use the identity principle.

\[ \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n + \sum_{n=0}^{\infty} nc_n x^n + 2 \sum_{n=2}^{\infty} c_{n-2} x^n + \sum_{n=0}^{\infty} c_n x^n = 0 \]

The common range is \( n \geq 2 \), so the terms corresponding to \( n = 0, 1 \) must be brought out.

\[ 2c_2 + c_0 + 6c_3 x + c_1 x + \sum_{n=0}^{\infty} [(n+2)(n+1)c_{n+2} + nc_n + 2c_{n-2} + c_n] x^n = 0 \]

The identity principle implies the following:

\[ c_2 = -\frac{c_0}{2} \]
\[ c_3 = -\frac{c_1}{3} \]
\[ c_{n+2} = -\frac{nc_n + 2c_{n-2} + c_n}{(n+2)(n+1)} = -\frac{c_{n+1} + 2c_{n-2}}{(n+2)(n+1)} \text{ for } n \geq 2 \]

From the given initial conditions, \( c_0 = 1 \) and \( c_1 = -1 \). The remaining coefficients are determined as follows:
\[ c_2 = -\frac{c_0}{2} = -\frac{1}{2} \]

\[ c_3 = -\frac{c_1}{3} = 1/3 \]

\[ n = 2 : c_4 = -\frac{c_2(3) + 2c_0}{12} = -\frac{-3/2 + 2}{12} = -\frac{1}{24} \]

\[ n = 3 : c_5 = -\frac{c_3(4) + 2c_1}{20} = -\frac{4/3 - 2}{20} = 1/30 \]

\[ n = 4 : c_6 = -\frac{c_4(5) + 2c_2}{30} = -\frac{-5/24 - 1}{30} = 29/720 \]

\[ n = 5 : c_7 = -\frac{c_5(6) + 2c_3}{42} = -\frac{1/5 + 2/3}{42} = -13/630 \]

\[ n = 6 : c_8 = -\frac{c_6(7) + 2c_4}{56} = -\frac{203/720 - 1/12}{56} = -143/40320 \]

So

\[ y = 1 - x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{24}x^4 + \frac{1}{30}x^5 + \frac{29}{720}x^6 - \frac{13}{630}x^7 - \frac{143}{40320}x^8 + \ldots \]

\[ y(0.5) \approx 0.4156 \]