Taylor series expansion of \( f = \cos(x) \) about \( x = 0 \).

\[
\begin{align*}
  f'(0) &= -\sin(0) = 0, & f''(0) &= -\cos(0) = -1 \\
  f^{(3)}(0) &= \sin(0) = 0, & f^{(4)}(0) &= \cos(0) = 1, & etc.
\end{align*}
\]

So Taylor series expansion is (as given in Problem 4.10)

\[
\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots
\]

An m-file that calculates this approximation with \( n \) terms is

```matlab
function apx=costaylor(x,n)
%Calculates the Maclaurin series approximation to \( \cos(x) \) using the first \( n \) terms in the expansion.
apx=0;
for i=0:n-1
    apx=apx+(-1)^i*x^(2*i)/factorial(2*i);
end
```

Problem 4.10 asks us to increment \( n \) from 1 until the approximate error indicates that we have accuracy to two significant digits for \( x = \pi/3 \).

We start with

\[
\begin{align*}
  &> a1=costaylor(pi/3,1) \\
  a1 &\approx 1 \\
  &> a2=costaylor(pi/3,2) \\
  a2 &\approx 0.4517
\end{align*}
\]

With \( \cos(\pi/3) = 0.5 \), the true error in \( a1 \) is 100% and in \( a2 \) it is

\[
true2=(a2-0.5)/0.5*100
\]
true2 =
-9.6623

That is about 9.7%. The approximate error, though is

>> aprerror=(a1-a2)/a2*100
aprerror =
121.3914%

With one more term we get

>> a3=costaylor(pi/3,3)
a3 =
0.5018

>> true3=(a3-0.5)/0.5*100
true3 =
0.3592

>> aprerror=(a2-a3)/a3*100
aprerror =
-9.9856

So the actual error is only 0.36% while the estimated is 10%.

Finally with a fourth term

>> a4=costaylor(pi/3,4)
a4 =
0.5000

>> true4=(a4-0.5)/0.5*100
true4 =
-0.0071

>> aprerror=(a3-a4)/a4*100
The true error is less than one percent of one percent, the approximate error is 0.37% so we have at least two significant digits and we stop.

Total error:

The difficulty of finding a good step size for differentiation is particularly acute when solving ill-conditioned equations. For example,

Consider the system:

\[10001u + xv = 1000,\]
\[xu + 10000v = 1000.\]

For \(x=10,000\) the determinant is almost zero. The system is ill conditioned, and it is hard to find a good step size for calculating derivatives with respect to \(x\).
One of my graduate students ran into difficulties calculating derivatives of the deformation of a car model seen below.
With respect to variables that define the dimensions of the car.