HOMEWORK 1:

Problem 1.6:

Derive the equation of motion of the compound pendulum consisting of uniform rod and disk.

Summing moments, we have:

\[ F d = I_O \ddot{\theta} \]  

where \( F \) is the moment force due to gravity, \( d \) is the distance to the center of mass, \( I_O \) is the inertia of the system about point \( O \), and \( \ddot{\theta} \) is the angular acceleration. The above equation will yield the equation of motion for the system. We just have to get expressions for the various components.

Finding the distance to the center of mass, \( d \):

\[ d = \frac{mL(\frac{L}{2}) + M(L + R)}{mL + M} \]  

Note: “m” is mass per unit length whereas “M” is just mass. “d” is the distance to the center of mass.
Finding the moment force due to gravity, $F$:

$$F = -W \sin(\theta) = -(mL + M)g \sin(\theta) \quad (3)$$

Finding the inertia of the system about point $O$. It should be noted that the inertia of the system is composed of the following three parts: the inertia of the rod about the end, the inertia of the disk about its center, and the parallel axis contribution for the disk. Therefore,

$$I_O = \frac{1}{3}mL^3 + \frac{1}{2}MR^2 + M(L + R)^2 \quad (4)$$

After substituting (2), (3), and (4) into (1) and rearranging, we arrive at:

$$\dot{\theta}(\frac{1}{3}mL^3 + \frac{1}{2}MR^2 + M(L + R)^2) + g(\frac{mL^2}{2}) + ML + MR)\sin(\theta) = 0$$

which is the equation of motion for the system.
Problem 1.8:
Derive the equation of motion of the system.

Translation of Point C:

\[-F - mg \sin(\theta) = m(R - r)\dot{\theta}\]

Rotation about Point C:

\[Fr = \frac{mr^2}{2} \ddot{\theta}\]

Relating the velocities, \(\dot{\theta}\) and \(\dot{\Psi}\):

\[v_c = (R - r)\dot{\theta} = r\dot{\Psi}\]

Therefore,

\[\dot{\Psi} = \frac{(R - r)}{r} \dot{\theta}\]
We can eliminate the contact force, $F$, from (5) by substituting the following obtained from (6):

$$F = \frac{mr^2}{2} \ddot{\Psi}$$

Also, from (7) we can see that

$$\ddot{\Psi} = \frac{(R-r)^2}{r} \ddot{\theta}$$

Substituting the above two equations into (5), we have:

$$m(R-r)\ddot{\theta} + \frac{m}{2}(R-r)\ddot{\theta} + mg\sin(\theta) = 0$$

Rearranging we have:

$$\ddot{\theta} + \frac{2g}{3(R-r)} \sin(\theta) = 0$$

which is the equation of motion for the system.
Problem 1.15:

Find the equivalent spring constant for the system.

From the book, we have that

\[ GJ(x) \frac{d\theta(x)}{dx} = M(x) \quad \text{with} \quad \theta(0) = 0 \quad (8) \]

Rearranging (8), we get:

\[ \theta(x) = M(x) \int_0^x \frac{d\eta}{GJ(\eta)} \]

Therefore,

\[ \theta(L) = \delta = M \int_0^L \frac{d\eta}{GJ[1 - \frac{1}{2}(\frac{\eta}{L})^2]} \]

The equivalent spring constant, \( k_{eq} \), is:

\[ k_{eq} = \frac{M}{\delta} = \frac{GJ}{\int_0^L \frac{d\eta}{[1 - \frac{1}{2}(\frac{\eta}{L})^2]}} \quad (9) \]

The main task is now simply performing the integration. To aid in the integration, let

\[ \eta = \sqrt{2}L \sin(\theta) \]

and hence

\[ d\eta = \sqrt{2}L \cos(\theta) d\theta \]

It should be noted that this change in variables requires a change in the limits of integration. Therefore, based on the following equation:

\[ \theta = \sin^{-1} \left( \frac{\eta}{\sqrt{2}L} \right) \]
The limits of integration become 0 to $\frac{\pi}{4}$.

After making the above substitutions and simplifying, the equation for $k_{eq}$ becomes

$$k_{eq} = \frac{GJ}{\sqrt{2L} \int_0^{\frac{\pi}{4}} \sec(\theta) d\theta}$$

Recalling that

$$\int \sec(\theta) d\theta = \ln |\sec(\theta) + \tan(\theta)| + C$$

We find that

$$k_{eq} = \frac{GJ}{\sqrt{2L} \ln(\sqrt{2} + 1)} \approx 0.8 \frac{GJ}{L}$$
Problem 1.20:

Find the equivalent inertia for the 2-gear system.

Note:
(1) Reaction forces on the gears at point of contact are equal in magnitude and opposite in direction.
(2) Angular acceleration of B is 'n' times that of A.

It should first be noted that:

\[
\frac{R_A}{R_B} = n
\]

To find the equivalent inertia of the system, I will apply an external moment, \( M_{ex} \), to Gear A and write torque equations for both gears. Therefore, after applying the external moment and recalling that the reaction force, \( F \), on the gears at the point of contact are equal in magnitude and opposite in direction, we have:

\[
\begin{align*}
\text{Gear A} & : \quad M_{ex} - FR_A = I_A \alpha_A \\
\text{Gear B} & : \quad FR_B = I_B \alpha_B \\
M_{ex} & = I_{eq} \alpha_A
\end{align*}
\]

Recalling that \( \alpha_B = n \alpha_A \) and using the equation for Gear B, we can solve for \( F \) as follows:

\[
F = \frac{n \alpha_A I_B}{R_B}
\]
Substituting (11) into the equation for Gear A in (10), we have:

\[ M_{ex} - \frac{n\alpha_A I_B R_A}{R_B} = I_A \alpha_A \]

which, after recalling the fact that \( \frac{R_A}{R_B} = n \) becomes

\[ M_{ex} - n^2 \alpha_A I_B = I_A \alpha_A \]

Substituting \( M_{ex} = I_{eq} \alpha_A \) into the above equation and simplifying, we find that the equivalent moment of inertia is

\[ I_{eq} = I_A + n^2 I_B \]