1. The string in the figure is in constant tension T. Derive the Lagrange equations of motion (i.e., from Hamilton’s principle) for the case of $m_1=m_2=m$, $L_1=L_2=L_3=L$ to obtain

$m\ddot{x}_1 + \frac{2T}{L} x_1 - \frac{T}{L} x_2 + mg = F_1$

$m\ddot{x}_2 - \frac{T}{L} x_1 + \frac{2T}{L} x_2 + mg = F_2$

The Lagrange equations are:

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}_k} \right) - \frac{\partial T}{\partial x_k} + \frac{\partial V}{\partial x_k} = F_k$$

Here $T = 0.5m\dot{x}_1^2 + 0.5m\dot{x}_2^2$, $\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{x}_k} \right) = m\ddot{x}_k$ and

$$V = mg(x_1 + x_2) + 0.5kx_1^2 + 0.5k(x_2 - x_1)^2 + 0.5kx_2^2 \quad k = T / L$$

$$\frac{\partial V}{\partial x_k} = mg + k(2x_k - x_{2-k})$$, which gives the desired equations of motion
2. Write the eigenvalue problem for the vibration frequencies and modes of the system in Problem 1 and use the Rayleigh quotient to estimate the frequencies. (You are not to obtain the results by solving the eigenvalue problem). Explain your choice of the modes used for the Rayleigh quotient.

The homogeneous equations are

\[ m\dddot{x}_1 + \frac{2T}{L} \ddot{x}_1 - \frac{T}{L} x_2 = 0 \]
\[ m\dddot{x}_2 - \frac{T}{L} \ddot{x}_1 + \frac{2T}{L} x_2 = 0 \]

Substituting \( x_i = u_i e^{i\omega t} \) we get

the vibration eigenvalue problem

\[ Ku - \omega^2 Mu = 0 \]

\[ K = k \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad M = m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

With only two degrees of freedom, and complete symmetry, the first mode should be \( [1 \ 1]^T \) and the second mode should be \( [1 \ -1]^T \)

So from the Rayleigh quotient

\[ \omega_1^2 = \frac{[1 \ 1]^T K [1 \ 1]}{[1 \ 1]^T M [1 \ 1]} = \frac{k}{m} = \frac{T}{Lm} \]
\[ \omega_2^2 = \frac{[1 \ -1]^T K [1 \ -1]}{[1 \ -1]^T M [1 \ -1]} = \frac{3k}{m} = \frac{3T}{Lm} \]
3. Answer the following questions in 20 words or less each
   a. What properties do the stiffness, and mass, matrices typically have in common, and what is usually different?

   They are both symmetric and semi-positive definite. The mass matrix is often diagonal and usually positive definite.

   b. What is the common denominator of algebraic and differential eigenvalue problems that is the characteristic of eigenvalue problems in general?

   These are homogenous sets of equations that for some values of a parameter (the eigenvalue) can have non-zero solution.

   c. Why it is a bad idea to solve problems of wave propagation by using modal expansion?

   Wave propagation problems often have discontinuities, and this causes modal expansion have convergence problems.