The Rayleigh Quotient

• First cover conversion of generalized eigen problem to standard one
  – Why bother?

• How do we estimate a frequency when we have estimate for the mode?
  – How do we know to estimate the mode?
  – The answer, the Rayleigh quotient is a wonderfully forgiving instrument
7:11 Converting to standard eigenproblem

- Software considerations sometimes make it desirable to convert vibration problem to standard eigenvalue problem.
- Use Cholesky decomposition.
- Define \( \mathbf{v} = \mathbf{L}^T \mathbf{u} \) then

\[
A \mathbf{v} = \lambda \mathbf{v} \quad A = L^{-1} KL^{-T} \quad \lambda = \omega^2
\]

- Why don’t we simply do

\[
M^{-1} \mathbf{Ku} = \omega^2 \mathbf{u}
\]
7.13: The Rayleigh quotient

- Given the vibration mode, we can calculate the vibration frequency as

\[ Ku_r = \omega_r^2 Mu_r \]

\[ \omega_r^2 = \frac{u_r^T Ku_r}{u_r^T Mu_r} \]

- As we will show, the Rayleigh quotient is useful for two reasons:
  - It is insensitive to errors in vibration mode
  - It is often possible to guess at the shape of a mode
R-Q using modal expansion

- Given an estimate $u$ to a vibration mode, expand $u$ in terms of a modal basis

\[
u = \sum_{r=1}^{n} c_r u_r = Uc
\]

\[
R = \frac{u^T Ku}{u^T Mu} = \frac{c^T U^T Ku c}{c^T U^T Mu c} = \frac{c^T \Omega c}{c^T c} = \sum_{r=1}^{n} \frac{\omega_r^2 c_r^2}{\sum_{r=1}^{n} c_r^2}
\]

- If $u$ is mostly in the shape of $u_r$, can easily show

\[
R = \omega_r^2 + \sum_{i \neq r}^{n} (\omega_i^2 - \omega_r^2) \varepsilon_i \quad \varepsilon_i = c_i / c_r
\]
Errors and application in optimization

- Error in R-Q, hence in frequency estimate proportional to square of error in mode
- For structural optimization we have good estimate for mode shape when we change a structure in the form of mode before change
- Can consider multiple possible changes without new eigensolution
- Similar to derivative approach, but not identical
Spring-mass example

- Estimate effect of increasing $k_4$ by 50%

- $m_1 = m_2 = m, m_3 = 2m, k_1 = k_2 = k_3 = k, k_4 = k_5 = k_6 = 2k.$

$$M = m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad K = k \begin{bmatrix} 2 & -1 & 0 \\ -1 & 6 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$
Matlab solution

\[ K = \begin{bmatrix} 2, & -1, & 0; \\ -1, & 6, & -1; \\ 0, & -1, & 3 \end{bmatrix}; \]
\[ M = \begin{bmatrix} 1, & 0, & 0; \\ 0, & 1, & 0; \\ 0, & 0, & 2 \end{bmatrix}; \]
\[ [V, D] = \text{eig}(K, M) \]

\[ K = \begin{bmatrix} 2, & -1, & 0; \\ -1, & 6, & -1; \\ 0, & -1, & 3.5 \end{bmatrix}; \]
\[ V'KV \]
\[ 1.5776 \quad 0.1134 \quad -0.0218 \]
\[ 0.1134 \quad 1.8333 \quad 0.0320 \]
\[ -0.0218 \quad 0.0320 \quad 6.3391 \]

Rayleigh Quotient

\[ [V, D] = \text{eig}(K, M) \]

\[ V = \begin{bmatrix} 0.3160 & -0.9223 & -0.2226 \\ 0.2083 & -0.1614 & 0.9647 \\ 0.6545 & 0.2484 & -0.0998 \end{bmatrix}; \]
\[ D = \begin{bmatrix} 1.3409 & 0 & 0 \\ 0 & 1.8250 & 0 \\ 0 & 0 & 6.3342 \end{bmatrix}; \]

Before

\[ V = \begin{bmatrix} -0.5309 & 0.8178 & -0.2221 \\ -0.2472 & 0.1012 & 0.9637 \\ -0.5732 & -0.4006 & -0.1050 \end{bmatrix}; \]
\[ D = \begin{bmatrix} 1.5343 & 0 & 0 \\ 0 & 1.8763 & 0 \\ 0 & 0 & 6.3394 \end{bmatrix}; \]

After
Error considerations

- Vibration mode is off by 10-70%, frequency is off by only 1.6%
- A designer, though, might say that the error is 21%
  - Predicted change in frequency: 1.2560 - 1.1580 = 0.098
  - Actual change in frequency is 1.2387 - 1.1580 = 0.081
- What units are used for the frequencies?
Reading assignment

Sections 7.15-7.16

Source: www.library.veryhelpful.co.uk/Page11.htm