Response to step, ramp and convolution

- Step function, integral of delta function
  - Forcing function often stepwise continuous
  - When can you also integrate the response

- Ramp function, integral of step function
  - Often serves same purpose as highway ramp
  - Building block
4.2 Step response

- Step function

\[ u(t - a) = \begin{cases} 
0 & \text{for } t < a \\
1 & \text{for } t > a 
\end{cases} \]

- Useful for notation. Instead of

\[ g(t) = \begin{cases} 
\frac{1}{m\omega_d} e^{-\zeta\omega_d t} \sin \omega_d t & \text{for } t > 0 \\
0 & \text{for } t < 0 
\end{cases} \]

write

\[ g(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_d t} \sin \omega_d t u(t) \]

\[ \delta(t - a) = \frac{du(t - a)}{dt} \]
Solution of step response

• Equation to solve

\[ \left( m \frac{d^2}{dt^2} + c \frac{d}{dt} + k \right) s = u(t) \]

• Solve from

\[ s(t) = \int_{-\infty}^{t} g(\tau) d\tau \]

• Obtain

\[ s(t) = \frac{1}{k} \left[ 1 - e^{-\zeta \omega_n t} \left( \cos \omega_d t + \frac{\zeta \omega_n}{\omega_d} \sin \omega_d t \right) \right] u(t) \]
Alternate derivation

• Differential equation
  \[ \left( m \frac{d^2}{dt^2} + c \frac{d}{dt} + k \right) s = u(t) \]

• General solution of homogeneous equation plus particular solution \( s = 1/k \)
  \[ s = \frac{1}{k} + e^{-\xi \omega_n t} \left( C_1 \cos \omega_d t + C_2 \sin \omega_d t \right) \]

• Find constant from
  \[ s(0) = \dot{s}(0) = 0 \]
  \[ s(t) = \frac{1}{k} \left[ 1 - e^{-\xi \omega_n t} \left( \cos \omega_d t + \frac{\xi \omega_n}{\omega_d} \sin \omega_d t \right) \right] u(t) \]
4.3 Ramp response

- Unit ramp function \( r(t-a) = (t-a)u(t-a) \)

- Relation to step function

\[
\begin{align*}
  r(t-a) &= \int_{-\infty}^{t} u(\tau - a) d\tau \\
  u(t-a) &= \frac{dr(t-a)}{dt}
\end{align*}
\]

- Differential equation for ramp response

- When does 2\textsuperscript{nd} equation hold?

\[
\begin{align*}
  \left( m \frac{d^2}{dt^2} + c \frac{d}{dt} + k \right) r &= u(t) \\
  r(t) &= \int_{-\infty}^{t} s(\tau) d\tau
\end{align*}
\]
Problem 4.5

- Find response of spring-mass-damper to \( F(t) \)

Solution:

\[
F(t) = \frac{F_0}{T} \left[ r(t) - r(t-T) \right]
\]

Then

\[
x(t) = \frac{F_0}{T} \left[ r(t) - r(t-T) \right]
\]

From problem 4.4 (homework)

\[
r(t) = \frac{1}{k} \left[ t - \frac{2\xi}{\omega_n} + e^{-\xi\omega_n t} \left( \frac{2\xi}{\omega_n} \cos \omega_d t + \frac{2\xi^2 - 1}{\omega_d} \sin \omega_d t \right) \right] u(t)
\]
Reading assignment

Sections 4.4, 4.5, 4.6

Source: www.library.veryhelpful.co.uk/ Page11.htm