WHAT IS THE AREA OF AN S-SIDED IRREGULAR POLYGON?

It is well known that any oblique triangle with specified vertices A, B, C has an area given by half the absolute value of the cross-product between two of its side-vectors. Alternatively one can use the Hero formula which requires only the side lengths to find the triangle area. In most cases the vector product approach is the simpler way to find the area. For example, consider a triangle with vertices at A[0,0], B[0,2], and C[1,1]. In this case the side-vector from A to B is \( \mathbf{BA} = 2 \mathbf{j} \) and the side-vector from A to C is \( \mathbf{CA} = \mathbf{i} + \mathbf{j} \). So evaluating \( \frac{|(\mathbf{BA} \times \mathbf{CA})|}{2} \) we find it equals 1 for the area of the isosceles triangle under consideration. To check this answer we can compare things with the Heron Formula:

\[
\text{Area} = \sqrt{s(s-a)(s-b)(s-c)} \quad \text{with} \quad s = \frac{(a+b+c)}{2} \text{ being the semi – perimeter }
\]

For the present case we have \( s=1+\sqrt{2} \) and \( \text{Area}=\sqrt{((\sqrt{2}+1)((\sqrt{2}-1)))}=1 \), which checks. It is our purpose here to extend the area calculations for a triangle to any larger S sided closed polygon of irregular side-length lying in the x-y plane.

In looking at S-sided polygons, one notices that they can always be broken up into S-2 contiguous triangles. So a rectangle can be represented as two triangles, a pentagon by 3 triangles, a hexagon by 4 triangles, and so on. That is, the number of triangles \( T=S-2 \). One also notices that the number of lines \( L \) required to form these sub-triangles is three less than the number of sides, so \( L=S-3 \). We summarize these geometric observations in the following figure-
Note that the rule we are using in drawing the red lines is always to connect those vertices separated from each other by two vertex points and that this continuous to hold when only red line polygons remain.

Let us next consider a specific area in the form of a pentagonal polygon which has straight lines connecting its five vertices located at A[0,0], B[1,0], C[2,2], D[1,3], and E[-1,2]. To determine the area of this polygon we begin with the following figure-

The figure is drawn to scale with $[x,y]$ representing the Cartesian coordinates of the vertices. Here $S=5$, $T=3$ and $L=2$. That is, we need to use only two division lines to form three sub-triangles. One possible configuration is as shown. We next write down the seven vectors defining all $S+L=7$ straight lines. These vectors are-

$\mathbf{BA} = \mathbf{i}$, $\mathbf{CB} = \mathbf{i} + 2\mathbf{j}$, $\mathbf{DC} = -\mathbf{i} + \mathbf{j}$, $\mathbf{ED} = -2\mathbf{i} - \mathbf{j}$, $\mathbf{AE} = \mathbf{i} - 2\mathbf{j}$, $\mathbf{CE} = 3\mathbf{i}$, and $\mathbf{CA} = 2\mathbf{i} + 2\mathbf{j}$

The vectors are indicated in bold lettering with the first letter referring to the head of the vector. Here $\mathbf{i}$ and $\mathbf{j}$ are unit length base vectors in the $x$ and $y$ direction, respectively. To find area 1 in the figure we write-

$\text{Area}_1 = |\mathbf{BA} \times \mathbf{CA}|/2 = |2\mathbf{k}|/2 = 1$

Next we find that area 3 is given by-

$\text{Area}_3 = |\mathbf{-CA} \times \mathbf{-CE}|/2 = |6\mathbf{k}|/2 = 3$

and area 2 by-

$\text{Area}_2 = |\mathbf{-ED} \times \mathbf{CE}|/2 = |3\mathbf{k}|/2 = 3/2$
This yields a total area of $11/2$ for the five-sided polygon. It is possible to break a polygon like this one into other triangles and possibly some other regular polygons. For instance in the above we have four right triangles (area = $1 + 1/2 + 1 + 1$) plus one rectangle (area = 2) giving us the same result of 5.5 for the area. Such an alternate breakup will however not be found for most irregular polygons. So it is best to stay with the present approach of breaking an irregular polygon into oblique sub-triangles. One should however always keep in mind any symmetry properties since these will reduce the number of required triangle area calculations.

You may have wondered what surveyors do when measuring the acreage of some land. It is essentially the same procedure as used above. They measure the length between neighboring vertices of the property and the angle of a north-south line relative to the line connecting neighboring vertices. With this information the area of the land using the above triangle breakup is readily determined. Typically the surveyors will give you a quote such as 5.3 acres more or less. This stems from the fact that most ground is not level and so a calculation carried out in one plane using measured field data will be slightly off.

The unit of land area in many English speaking countries is still the acre which equals exactly 43,560 sq.ft. It is $1/640^{th}$ of a section which equals one square mile. A township equals 36 sections or 23040 acres. The acre was the unit of land a typical farmer in the middle ages could plow in one day using two oxen. The word origin is from the German for plowing which is ackern. During the US Homestead Acts of the 18 hundreds land in the size from quarter to full sections was made available by the US government to any head of household demonstrating that he can successfully farm the land for a period of five years. Today when flying over the mid-west one often sees peculiar green circles of half-mile diameter. These are the result of central pivot irrigation centered on quarter section (160 acres) plots many on land acquired by the early homesteaders. At the rate of today’s $10,000$ dollars per acre for good farmland, the offspring of many of these early nearly penniless homesteaders are now quite wealthy.

The metric measure for land area is the hectare which is a piece of land of 100 x 100 meter. The conversion to English units is 1 hectare = 2.47105 acre. At my wife’s birthplace in Bavaria they also use the archaic area measure of “Tagwerk” (translation – days work) which equals approximately 40,000 sq.ft and thus has the same agricultural origin as the English definition for the acre.

Let now continue on with some more area calculations. Consider a regular eight-pointed star whose eight outer vertices lie on a circle of radius $R=3$ and inner eight vertices lie on a circle of radius $R=1$ as shown-
This is a very easy problem to analyze when making use of its eight-fold symmetry. Following our rule of connecting the vertices \( n \) with \( n+2 \) we get 8 isosceles triangles and a central regular octagon. Next connecting every second vertex of the octagon we get 4 more triangles plus a central square which can be divided into two equal triangles as shown. The results are \( 8+4+2=14 \) isosceles triangles and \( 8+4+1=13 \) red lines needed to construct these. The total number of sides of the star is 16. Note that there are other ways to divide up the star into sub-triangles, but these would violate the condition that \( T=S-2 \) and \( L=S-3 \). Fortunately here we don’t need to run through an evaluation of the areas of each of the 14 triangles in order to find the star area. The much simpler approach is to note that all 16 vertices are given in cylindrical coordinates by-

\[
(\rho, \theta) = (2, \frac{n\pi}{2}) \]

So we need to only look at the area of the single triangle with vertex points at \( a[0,0] \), \( b[3,0] \) and \( C[\cos(\pi/8),\sin(\pi/8)] \) and multiply the result by 16 to find the area of the star. The area of the triangle is simply –

\[
\text{Area}_{\text{triangle}} = |(3i) \times [(i\cos(\pi/8)+j\sin(\pi/8))]|/2 = (3/2)\sin(\pi/8)
\]

so that the area of the star becomes \( 24\sin(\pi/8) = 8.306809 \ldots \) This example clearly shows that the evaluation of polygon areas should always be looked at from a standpoint of symmetry.
As a last example, look at the following thirteen sides polygon and its eleven sub-triangles –

Here we see that all but the one triangle C-D-E have an area of 1 each. The area of C-D-E however is:

\[ \text{Area}_{C-D-E} = \frac{|(-2i+j) \times (-i-j)|}{2} = \frac{3}{2} \]

Thus the total area of the 13 sided irregular polygon becomes 23/2.