GENERATING 2D CURVES USING THE FIRST ORDER ODEs dy/dx=f(x,y) or dr/rdθ=g(r,θ)

The first order ODEs dy/dx=f(x,y) or dr/rdθ=g(r,θ) lend themselves to graphical solutions having slopes of either f(x,y) or g(r,θ). There are an infinite number of such solutions with many of the curves having the names of the mathematician who first studied them. We want in this article to generate many of the better known curves attached to specific values of f(x,y) or g(r,θ).

Let us begin with dy/dx=xexp(y) subject to y(0)=0. Separating the variables , we have-

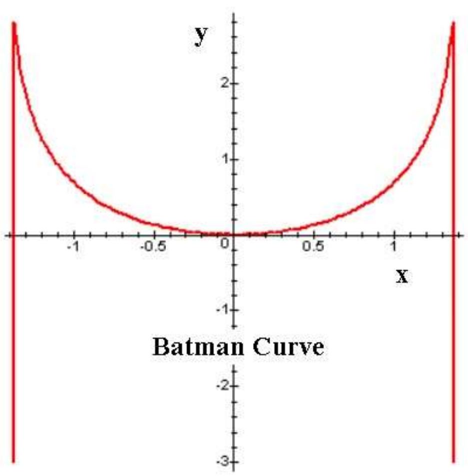
dy/exp(y)=xdx

On solving and applying the y(0)=0 condition we find the curve-

exp(y)=1/(1-x^2/2)

An implicit plot looks like this-

dy/dx=x exp(y) subject to y(0)=0

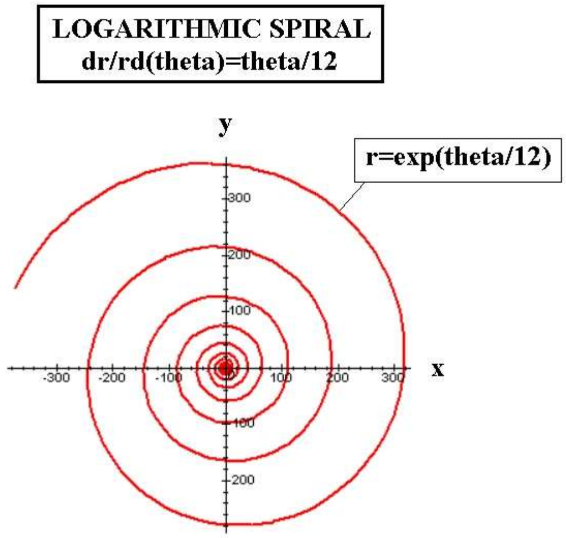


This curve has no name attached to it, so I choose to call it the Batman Curve due to the facial resemblance.

Next we look at the equation dr/rdθ=a subject to r(0)=1. The solution in polar form becomes-

r=exp(aθ)

This curve was studied in great detail by Jacques Bernoulli and is often referred to as the Logarithmic Spiral or Spira Mirabilis. Bernoulli was so proud of his discovery that he had it engraved on his tombstone in Basel, Switzerland. During my Fulbright year in Freiburg, Germany back in 1961-62 I had a chance to visit Bernoulli's grave a few miles down the road. A graph of his spiral when a=1/12 follows-



Note that in generating such curves from first order equations one can use either Cartesian or polar coordinates. As to which is better can be determined by looking at both possibilities.

Consider next g(r,θ)=sin(θ)/r or its equivalent f(x,y)=(4x-1-x/sqrt(x^2+y^2))/(-4y+4/sqrt(x^2+y^2). Here clearly the slope g(r,θ) is the simpler to solve. Doing so with r(0)=0 yields the closed form solution and curve of-

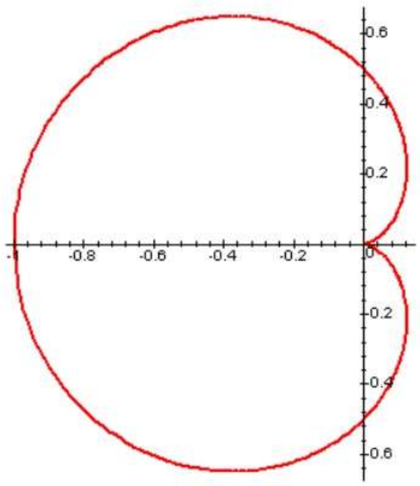
r=0.5[1-cos(θ)]

This curve is the famous cardioid. In Cartesian Coordinates it has the more complicated form-

[sqrt(x^2+y^2)-x]=2(x^2+y^2)

Its graph expressed in either coordinate system has the same form-

CARDIOID r=0.5[1-cos(theta)]



r(0)=0, r(Pi/2)=0.5, r(Pi)=-1

Next we examine the ODE dy/dx=x^(2/3)+y^(2/3) with y(1)=0. After some manipulations one finds the closed form solution-

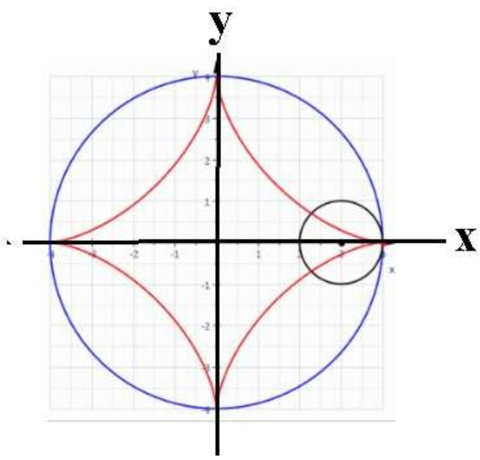
x^(1/3)+y^(1/3)=1

whose graph looks like this-

ASTROID

(Four Cusp Hypocycloid)

dy/dx=-(y/x)^(2/3)



circle diameter ratio equals 4 to 1

Note that the Astroid represents the path of a point on the periphery of a smaller circle of radius r rolling around the inside of a larger circle of radius R. To have a cusp at four equally spaced points along the outer circle means $2\pi r=2\pi R/4$. That is, $4r=R$.

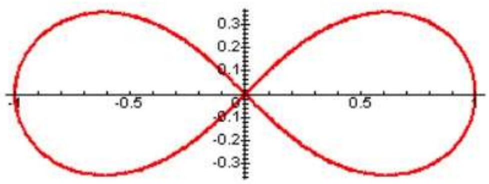
Another interesting slope $dr/(rd\theta)=-\sin(2\theta)/r^2$ with $r(0)=1$, produces the solution-

$$r^2=\cos(2\theta)$$

which produces what is known as the Lemniscate-

LEMNISCATE

dr/(rd(theta)=-sin(2 theta)/r^2



r^2=cos(2 theta)

You will note that this figure looks identical to the standard sign for infinity. In Cartesian Coordinates it reads-

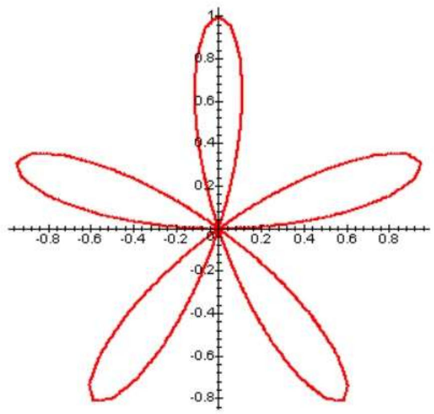
(x^2+y^2)^2=(x^2-y^2)

Another important curve has the slope dr/[r d(theta)]=5cos(5 theta)/r .It solves as-

r=sin(5theta)

and is known as a five petal Rhodonea Curve. It looks as follows-

FIVE PETAL RHODONEA
r=sin(5theta)



In Cartesian Form:
r^6=5yr^4-20y^3r^2+16y^5   with r=sqrt(x^2+y^2)

The polar form of the graph has a much simpler appearance than its equivalent Cartesian form given at the bottom of the graph. It also is observed to have the shape of a typical home ceiling fan.

Consider next the equation for a circle of radius R=3 centered at [x,y]=[2,3]. It is governed by the equation-

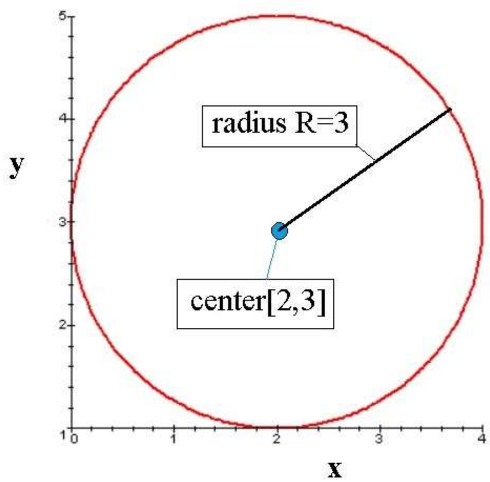
f(x,y)=-dy/dx=-(x-2)/(y-3) subject to y(2)=0

Using the substitution X=x-2 and Y=y-3, we find the solution-

(x-2)^2+(y-3)^2=3^2

This indeed represents an off-center circle of radius R=3 centered at [x,y]=[2,3]. Here is the graph-

CIRCLE (x-2)^2+(y-3)^2=3^2



f(x,y)=-(x-2)/(y-3)=3^2 with y(2)=0

Another interesting first order ODE is-

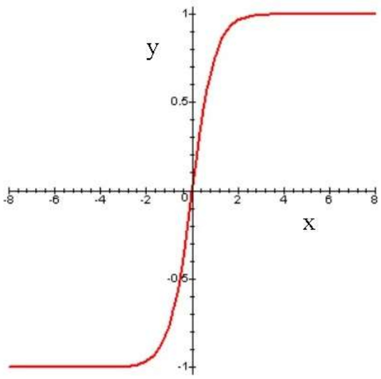
dy/dx=1/(cosh(x))^2=4/(exp(2x)+2+exp(-2x)) subject to y(0)=0

On integrating we find –

y=tanh(x)

This produces the identities y(-∞)=-1, y(0)=0 and y(+∞)=+1.Its graph looks as follows-

Y=TANH(X)



f(x,y)=dy/dx=1/[cosh(x)]^2

While I was a graduate student at Princeton over sixty years ago we where using this type of curve to describe the difference in gas density between two sides of a shock wave in air.

As a final and one of the simplest graphs consider one whose slope in polar coordinates reads g(r,θ)=a/r with r(0)=0. It is governed by the first order ODE-

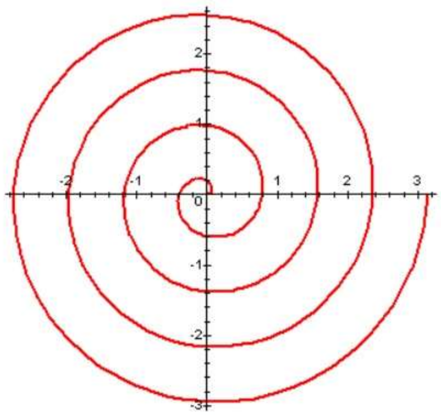
$$dr/d\theta = a \text{ subject to } r(0)=0$$

which solves as –

$$r=a\theta$$

and is known as the Archimedes Spiral. Its graph for a=1/8 follows-

ARCHIMEDES SPIRAL



r=a theta   a=1/8

Although it has a very simple analytic form in polar coordinates, the actual length S of the spiral after N turns has a rather complicated form. We find

$$S=a\int_0^{b=2\pi N}\sqrt{1+\theta\hat{}2}\,d\theta=\left(\frac{a}{2}\right)\{b sqrt(1+b^2)+\ln[b+\ sqrt(1+b^2)\}$$

with b=$2\pi N$.

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Nov.9, 2020
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