SOLUTION OF SIMULTANEOUS DIOPHANTINE EQUATIONS

A Diophantine Equation is an expression involving two variables x and y taken to specified powers and added or multiplied. Only integer values for x and y are allowed. The simplest set of two simultaneous Diophantine equations is the linear form-

ax+by=c with dx+ey=f

with a,b,c,d,e,and f as specified integers. By eliminating either x or y, these two equations have the closed form solution-

$$x=\frac{(bf-ce)}{(bd-ae)} \qquad \text{and} \qquad y=\frac{(cd-af)}{(bd-ae)}$$

These values can also be obtained by solving the matrix equation-

$$\begin{bmatrix} a & b \\ d & e \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix}$$

To have a bounded solution it is necessary that the determinant of the 2x2 matrix shown does not vanish.

We want in this note to consider more complicated pairs of Diophantine Equations in which x and y are taken to specified powers different from one.

We begin by looking at the non-linear Diophantine Pair -

a sqrt(x)+by=c and dx +e sqrt(y)=f

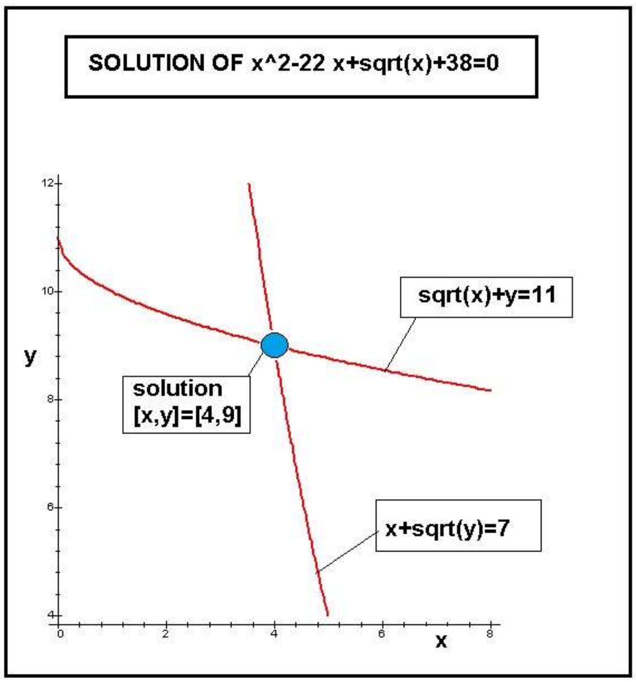
These cannot be solved by matrix methods but can be found by elimination. From the first we have y=[c-a sqrt(x)]/b. Plugging into the second produces-

x^2-Ax+Bsqrt(x)=C

, where A=2f/d , B=(a/b)(e/d)^2, and C=(ce^2-bf^2)/(bd^2). Since sqrt(x) must be an integer, we know x has one of the values 0,1,4,9,16,…Also A,B,and C must be integers. For specified positive or negative integer values of A, B, and C, the equation in x can be solved. We are looking for those solutions where x is equal to the square of an integer. One case for which his works is A=14, B=1, and C=38.It produces-

sqrt(x)+y=11 and x+sqrt(y)=7

and solves as x=4 and y=9. A graphic solution is also possible as shown-



Consider next the non-linear simultaneous Diophantine Equations-

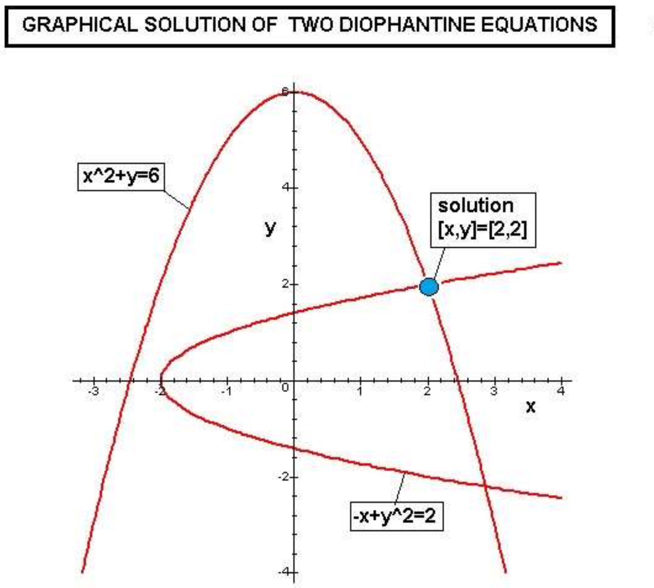
x^2+y=6 with x+y^2=2

These can be converted to the quartic algebraic equation-

y^4-4y^2+y=2

By inspection one sees at once that y=2 satisfies the equation meaning x=2 also.by looking at the original pair one sees that the integer solution

[x,y]=[2,2] is given by the intersection of two parabolas. Here is the graph-



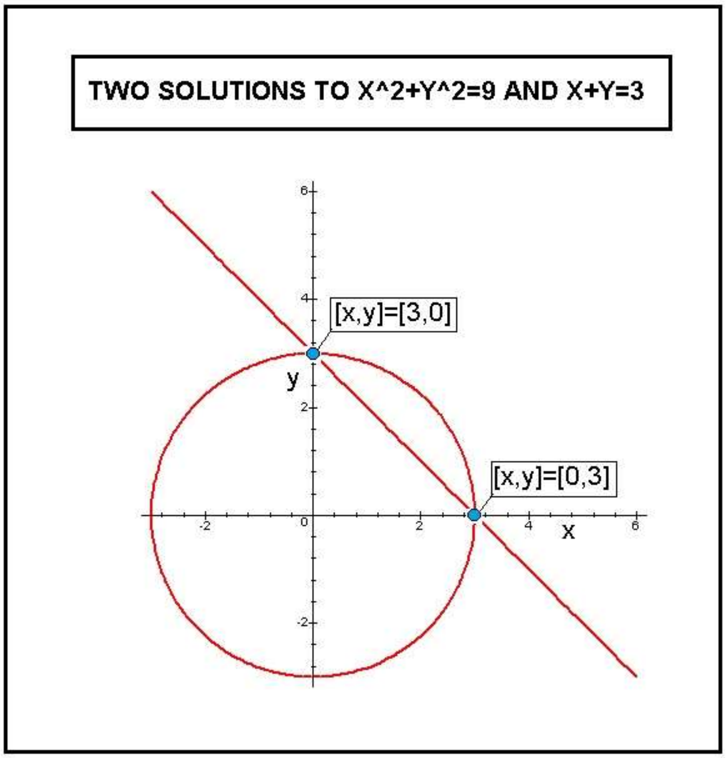
Next consider the Diophantine Pair-

x^2+y^2=c and x+y=k

, where c and k are specified constants.The first equation represents a circle of radius r=sqrt(c) and the second a straight line with slope of minus one and value y=k at x=0. Eliminating x or y produces the solutions-

x=(k^2-c)/2k and y=(k^2+c)/2k

To be valid solutions we must have k unequal to zero and the two quotients to be integers.When k=3 and c=9, we get the two Diophantine solutions [x,y]=[3,0] and [0,3]. A graphical representation of the two integer solutions follow-



As a final example consider the simultaneous equations-

(p+q)/2=S and pq=N

Here p and q are the unknown prime factors of the semi-prime N=pq and S is the mean value of the two prime components. Solving for p and q , we get-

p=S-sqrt(S^2-N) and q=S+sqrt(S^2-N)

Since both p and q are required to be integers, it is sufficient to have sqrt(S^2-N) be an integer.

Consider the semi-prime N=77. Here S=9 produces the radical 2 and we get p=9-2=7 and q=9+2=11. The factoring of large semi-primes N=pq plays a significant role in connection with public key cryptography. If one knows the value of the sigma function encountered in number

theory, the value of S can also be written as S=(σ-1-N)/2. So the semi-prime N=77 has the sigma function σ=96 leaving S=(96-1-77)/2=9.

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